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Group theoretical approach to quantum fields in de Sitter space I. The principal series

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Abstract: Using unitary irreducible representations of the de Sitter group, we construct the Fock space of a massive free scalar field. In this approach, the vacuum is the unique dS invariant state. The quantum field is *a posteriori* defined by an operator subject to covariant transformations under the dS isometry group. This insures that it obeys canonical commutation relations, up to an overall factor which should not vanish as it fixes the value of $\hbar$. However, contrary to what is obtained for the Poincaré group, the covariance condition leaves an arbitrariness in the definition of the field. This arbitrariness allows to recover the amplitudes governing spontaneous pair creation processes, as well as the class of alpha vacua obtained in the usual field theoretical approach. The two approaches can be formally related by introducing a squeezing operator which acts on the state in the field theoretical description and on the operator in the present treatment. The choice of the different dS invariant schemes (different alpha vacua) is here posed in very simple terms: it is related to a first order differential equation which is singular on the horizon and whose general solution is therefore characterized by the amplitude on either side of the horizon. Our algebraic approach offers a new method to define quantum field theory on some deformations of dS space.

Keywords: Space-Time Symmetries, Global Symmetries, dS vacua in string theory.

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Contents

1. Introduction

In flat spacetime, there are several equivalent ways to quantize a massive free scalar field. One can start with the invariant action, impose the canonical commutation relations, and derive the generators of the Poincaré group as Noether charges [1]. From a Hamiltonian point of view, one has a collection of harmonic oscillators labelled by the spatial momentum. One then verifies that the one-particle sector of the Fock space of these oscillators is a unitary irreducible representation (UIR) of the Poincaré group [2].

Alternatively, one can start from the UIR of the Poincaré group and construct a Fock space as the direct sum of the symmetrized tensor product of the UIR Hilbert space [3, 4]. One can then introduce creation and annihilation operators which are associated to a basis in the UIR Hilbert space, and which are linearly combined to form a local field. The latter is uniquely defined by the requirement that it should covariantly transform:

$$\Phi(\Lambda x) = U(\Lambda)\, \Phi(x)\, U^{\dagger}(\Lambda)\,, \tag{1.1}$$

where $U(\Lambda)$ is the unitary operator representing the Poincaré transformation Λ in the Fock space. This procedure is explained in more detail in Section 2.

Thirdly, one can quantize a massive scalar field from an analysis of Green functions $G(x;x')$, invariant solutions of the Klein-Gordon equation with well defined boundary conditions. Explicitely, for the positive frequency Wightman function, one writes

$$G_+(x;x') = \sum_n \phi_n(x)\,\phi_n^*(x')\,, \tag{1.2}$$

where $\{\phi_n, \phi_n^*\}$ form a complete set of orthogonal solutions (of norm $+1, -1$) with respect to the Klein-Gordon product. One can then define the field operator as

$$\Phi(x) = \sum_n \phi_n(x)\,a_n + \phi_n^*(x)\,a_n^\dagger\,. \tag{1.3}$$

The equal time canonical commutation relations are then equivalent to

$$[\,a_n,\, a_{n'}^\dagger\,] = \delta_{n,n'}\,, \qquad [\,a_n,\, a_{n'}\,] = 0\,. \tag{1.4}$$

In this approach the modes ϕ_n are univocally defined by the positive frequency condition implied by eq.(1.2).

When quantizing fields in dS space, one encounters novel features which engender difficulties [5, 6, 7, 8, 9, 10, 11, 12] and which are deeply rooted in the absence of a time-like Killing vector field which is globally defined, see Figure 1.

The approach which has been mostly followed is the third one we just mentioned. It has been realized that there exists a two-parameter family of dS invariant Green functions (i.e. functions of a quantity which coincides with the dS invariant geodesic distance whenever the latter exists between the two points under consideration).

The physical interpretation of the degeneracy is provided in terms of different vacuum states related to each other by Bogoliubov transformations [11]. These states are generally referred as *alpha vacua*.

An additional important notion is provided by the behavior of the Green function in the coincidence point limit. When requiring that this behavior be that of Hadamard [13], which is a covariant way to impose vacuum condition in the flat space-time limit, only one alpha vacuum is picked out. It coincides with the so-called Bunch-Davies (BD) vacuum defined by a positive conformal frequency condition in the high momentum limit [8]. There are two interesting aspects related to this state which are worth mentioning. First, the BD vacuum is perceived by an inertial particle detector as a thermal bath at a temperature given by $H/2\pi$ where H is the Hubble parameter. This is in agreement with the dS horizon temperature derived from the first law of thermodynamics and the Bekenstein-Hawking entropy [9] associated to the area of the cosmological horizon. The second aspect arises from the fact the initial vacuum, conventionely defined by the absence of massive quanta as seen by an inertial particle detector at asymptotic early times, neither coincides with the BD vacuum nor does it with the final vacuum [10]. This vacuum instability results from the spontaneous creation of pairs of massive particles in dS space. One thus faces a dilemma: either one insists on the Hadamard character and one works with the BD vacuum which contains coherent superpositions of pairs of particles at early times, i.e. a state which cannot be prepared at early times, or one gives up the Hadamard condition and one deals with a state whose energy density is infinite using the standard renormalization procedure. The only way to get states which are both Hadamard and contain no particle at early times is to break the dS invariance of the state.

In this paper, we reconsider the quantization of a massive scalar field from a group theory point of view. Our approach is close to the second method in the above list. Namely, we start with the UIR of dS group [22]-[30], that is, $\mathrm{SO}(1,n)$ where n is the space-time dimension. The representations fall into three different classes called the principal, the complementary, and the discrete series [30]. Each series requires a separate analysis. In this paper we study the principal series, the other two series will be presented in a separate paper.

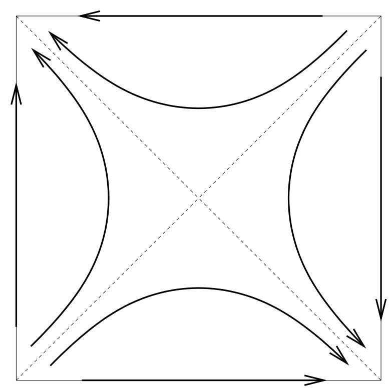

Figure 1: The flow of a boost Killing vector field is represented in a Carter-Penrose diagram of de Sitter space. The bifurcating horizon is the locus where the null horizons meet. The Killing vector field is time-like and future directed only in the left quadrant, i.e. for the points which are space-likely separated from the bifurcating horizon, and on its left. The vector fields of the other isometries of dS space are all space-like.

The representations belonging to the principal series are univocally given in terms of the UIR of the *compact* group $\mathrm{SO}(n-1)$ and a real parameter which fixes the quadratic Casimir together with the mass of the field. From the Hilbert space of this UIR and the trivial representation, we build a Fock space wherein there is a unique dS invariant vacuum state. In this construction therefore, we do not recover the above mentioned two-parameter ambiguity of the dS invariant vacua. In fact, it is our aim to discover how will this ambiguity reappear when constructing the local field from the UIR. It is also our endeavor to understand how the notion of vacuum instability translates in terms of group representations. Both questions are explicitly answered by the construction we present.

To construct the Fock space we start with the trivial representation and the Hilbert space carrying the UIR. Explicitly, the vacuum corresponds to the trivial representation, the one particle sector to the UIR, and the n particles sectors are given by symmetrized tensor products of the latter Hilbert space. We can then define creation and annihilation operators, and realize the generators of the dS group on this Fock space as bilinear operators in a and $a^\dagger$. This structure guarantees that each sector with a given particle number remains invariant under dS transformations. Moreover if interactions among sectors are expressed in terms of these operators there is still no ambiguity in the quantum theory. In fact it is only through the requirement that the interactions be local in space-time that the ambiguity reappears. Indeed, when constructing local field operators from the UIR, we get a family of covariant and canonical fields parameterized by elements of $\mathrm{SU}(1,1)/\mathrm{U}(1)$. Two members of the family are formally related by an infinite product of (two-mode) squeezing operators which commutes with all generators of dS group. Contact with the usual treatment is made by computing the Green functions associated with the various field operators. In a nutshell one has the following equalities:

$$\begin{aligned} G_{\alpha,\beta}(x,x') &= \langle\Omega|\,\Phi_{\alpha,\beta}(x)\,\Phi_{\alpha,\beta}(x')\,|\Omega\rangle \\ &= \langle\Omega|\,\mathcal{S}^{\dagger}_{\alpha,\beta}\,\Phi(x)\,\Phi(x')\,\mathcal{S}_{\alpha,\beta}|\Omega\rangle \\ &= \langle\Omega_{\alpha,\beta}|\,\Phi(x)\,\Phi(x')\,|\Omega_{\alpha,\beta}\rangle\,, \end{aligned} \tag{1.5}$$

where $|\Omega\rangle$ is the unique vacuum state, $\Phi_{\alpha,\beta}$ is the local field labelled by (α,β), which are coordinates of $\mathrm{SU}(1,1)/\mathrm{U}(1)$, and $\mathcal{S}_{\alpha,\beta}$ is the squeezing operator relating this field to a reference one, e.g. characterized by $(0,0)$. In the third line $|\Omega_{\alpha,\beta}\rangle$ is the corresponding alpha vacuum as usually defined. The correspondence between the two treatments is made unambiguous by the identifying the Green function associated with the BD vacuum in both of them.

Besides offering an interesting alternative way to recover known results, the present approach possesses several advantages with respect to the usual approach. For instance, the various solutions for the field operator arise from a first order differential equation which is singular on the cosmological horizon. This offers a simple and geometrical interpretation of the solutions in terms of the field amplitude on either side of the horizon. In particular it offers a straightforward identification of the three most relevant cases, the BD, the initial and final vacua. In this respect, we point out that the quantization of a scalar field in a constant electric field is also efficiently obtained by a similar treatment based on a first order differential equation which is singular on the (acceleration) horizon [32, 33]. In that case as well, the amplitudes of the solutions on either side directly govern the pair creation probability amplitudes.[1]

An additional advantage of the group theory approach is that explicit solutions of the field operator are obtained from the first order equation without having to solve the Klein-Gordon equation. In a similar vein, Green functions are given by well defined integrals. In addition, these include the $i\epsilon$ prescription encoding the holomorphic properties of the modes on the horizon.

Moreover, given that the isometry groups and therefore the representations of dS and AdS in (1+1)-dimensions are identical (modulo the identification of the mass square), the local field we constructed can be viewed as living on AdS upon exchanging temporal and spatial coordinates and momenta. The principal series here considered describes tachyonic scalar fields in AdS.

Finally, we hope that the present algebraic approach to quantum field theory on dS space will offer interesting applications when considering deformations of the dS group, which could result from some noncommutative description of spacetime [35] or perhaps also from self gravitational effects in (2+1)-dimensions [36]. In particular, the calculation of the modified pair creation amplitudes could proceed along the same lines as the ones we adopted.

[1] The possibility of re-expressing, in the presence of Killing horizons, second order differential equations by singular first order equations which encode the holomorphic character of the positive frequency modes in an interesting feature which deserves further study as it leads to horizons thermodynamics [34] and encompasses in a unified treatment, electro-production, Hawking effect, Unruh effect, and pair creation in de Sitter space.

The paper is organized as follows. In Section 2, we present our method by considering the quantization of a massive scalar field in a flat two-dimensional spacetime. We review in Section 3 the basic properties of the representations belonging to the principal series of the dS group in two dimensions. In Section 4, we construct the local field, verify that it automatically obeys canonical commutation relations and study the effect of time reversal. In Section 5 we compute the Wightman function. This allows to identify the particular field giving rise to the BD vacuum. All the other solutions can then be expressed in terms of elements of $\mathrm{SU}(1,1)/\mathrm{U}(1)$. This offers a direct comparison with the parameterization of the Green functions given by Allen [11] and therefore to make contact with the class of alpha vacua. To complete the quantization we study, in Section 6, the time evolution of the field operator, we identify the initial and final positive frequency modes, and compute the pair creation probability amplitudes. In Section 7, we construct the squeezing operator relating couples of alpha vacua in the usual approach, and pairs of canonical fields in our treatment. Finally, in Section 8 we generalize this algebraic approach to the n-dimensional case and show that the moduli space of canonical fields remains $\mathrm{SU}(1,1)/\mathrm{U}(1)$. In Appendix A, we compute the Wightman functions in terms of the holomorphic and anti-holomorphic solutions of the above mentioned first order differential equation. In Appendix B, we use our algebraic formulation to describe the quantization of the scalar field defined on the flat sections of dS space.

2. Massive scalar field in Minkowski spacetime

In order to illustrate the method we shall use, let us first consider a massive scalar field in the two-dimensional flat space-time. We shall show how starting from a Unitary Irreducible Representation (UIR) of the Poincaré group, we can construct the corresponding canonical scalar field operator.

The Poincaré group is generated by a space translation operator, a time translation and a boost. When these operators act on the UIR, they are respectively noted by $\mathcal{P}$, $\mathcal{H}$ and $\mathcal{K}$. The algebra is

$$[\mathcal{P},\,\mathcal{H}] = 0\,, \qquad [\mathcal{K},\,\mathcal{P}] = i\mathcal{H}\,, \qquad [\mathcal{K},\,\mathcal{H}] = -i\mathcal{P}\,. \tag{2.1}$$

The irreducible scalar representations of the Poincaré group are characterized by the value of the quadratic Casimir $\mathcal{C} = -\mathcal{H}^2 + \mathcal{P}^2 = M^2$ and the sign of the energy. At fixed M, the representation is realized on the Hilbert space $\mathscr{H}$ with basis $|\mathbf{p}\rangle$, and scalar product $\langle\mathbf{p}'|\mathbf{p}\rangle = \delta(\mathbf{p}-\mathbf{p}')$ as

$$\begin{aligned} \mathcal{P}\,|\mathbf{p}\rangle &= \mathbf{p}\,|\mathbf{p}\rangle\,, \qquad \mathcal{H}\,|\mathbf{p}\rangle = \omega(\mathbf{p})\,|\mathbf{p}\rangle\,, \\ \mathcal{K}\,|\mathbf{p}\rangle &= \frac{i}{2}\left\{\frac{\partial}{\partial\mathbf{p}},\,\omega(\mathbf{p})\right\}\,|\mathbf{p}\rangle\,, \end{aligned} \tag{2.2}$$

where $\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + M^2}$ is the positive root of $\mathcal{C} = M^2$.

From the Hilbert space $\mathscr{H}$ and the trivial representation of the Poincaré group of Hilbert space $\mathscr{H}_0$ (of dimension one and spanned by the vector $|\Omega\rangle$), we construct the Fock space as

$$\mathscr{F} = \mathscr{H}_0 \oplus \bigoplus_{n=1}^{\infty} \bigotimes_{s}^{n} \mathscr{H}\,, \tag{2.3}$$

where the n^{th} term is the representation obtained by taking the symmetrical tensor product of n copies of the UIR. The Fock space carries a reducible unitary representation of the Poincaré group. The generators will be noted P, H and K, and the group elements $U(\Lambda)$. We also introduce creation and annihilation operators and denote them $a^\dagger(\mathbf{p})$ and $a(\mathbf{p})$. They obey

$$\begin{aligned} a^\dagger(\mathbf{p})|\Omega\rangle &= |\mathbf{p}\rangle\,, \\ \Big[a(\mathbf{p}),\, a^\dagger(\mathbf{p}')\Big] &= \delta(\mathbf{p}-\mathbf{p}')\,, \end{aligned} \tag{2.4}$$

where $|\Omega\rangle$ is annihilated by all the annihilation operators $a(\mathbf{p})$ and is thus the vacuum of the Fock space. More explicitely the operators acting on Fock space are

$$U(\Lambda) = \int d\mathbf{p}d\mathbf{p}'\, \langle \mathbf{p}|\,\mathcal{U}(\Lambda)\,|\mathbf{p}'\rangle\, a^\dagger(\mathbf{p})\, a(\mathbf{p}')\,, \tag{2.5}$$

where $\mathcal{U}(\Lambda)$ acts on the UIR.

Our goal now is to construct a *local* field operator on $\mathscr{F}$, $\Phi(x)$, which is a linear superposition of creation and annihilation operators. The basic requirement on this local field is the covariance transformation property eq.(1.1). This implies that the field is determined by its value at the origin:

$$\Phi(x) = e^{-ix\cdot P}\,\Phi(0)\,e^{ix\cdot P}\,, \tag{2.6}$$

where $x\cdot P = -tH + \mathbf{x}P$. It also implies that $\Phi(0)$ is invariant under the boost transformation which leaves the origin fixed:

$$[\,K,\,\Phi(0)\,] = 0\,. \tag{2.7}$$

Writing $\Phi(0)$ as

$$\Phi(0) = \int d\mathbf{p}\,\Big(\,c(\mathbf{p})a^\dagger(\mathbf{p}) + c^*(\mathbf{p})a(\mathbf{p})\Big)\,, \tag{2.8}$$

eq.(2.7) implies that

$$\omega(\mathbf{p})c'(\mathbf{p}) + \frac{1}{2}c(\mathbf{p})\omega'(\mathbf{p}) = 0\,, \tag{2.9}$$

which gives

$$c(\mathbf{p}) = \frac{A}{\sqrt{\omega(\mathbf{p})}}\,, \tag{2.10}$$

where A is a constant complex amplitude. It should be noticed that $c(\mathbf{p})$ is even in $\mathbf{p}$. This implies that $\Phi(0)$ is left invariant under parity. It is also interesting to study the behavior of $\Phi(0)$ under time reversal $t \to -t$ which is the other global symmetry which leaves $(0,0)$ invariant. It is implemented by the (anti-unitary) operator satisfying

$$\mathsf{T}\,H\,\mathsf{T}^{-1} = H\,, \qquad \mathsf{T}\,P\,\mathsf{T}^{-1} = -P\,, \qquad \mathsf{T}\,K\,\mathsf{T}^{-1} = K\,. \tag{2.11}$$

When acting on the field operator one has

$$\mathsf{T}\,\Phi_{\{A\}}(t,\mathbf{x})\,\mathsf{T}^{-1} = \Phi_{\{A^*\}}(-t,\mathbf{x})\,, \tag{2.12}$$

where we have written A as a subscript to make explicit that the field depends on it.

It should be emphasized that the linear scalar field Φ, the solution of eq.(1.1) automatically satisfies the equal time commutators. Indeed, one has

$$[\,\Phi(0,0),\ \Phi(0,\mathbf{x})\,] = 0\,, \tag{2.13}$$

and

$$[\,\partial_t\Phi(0,0),\ \Phi(0,\mathbf{x})\,] = -4\pi i|A|^2\delta(\mathbf{x})\,. \tag{2.14}$$

The latter has the required δ dependence, and can thus be used to fix the norm of A in terms of $\hbar$, namely $4\pi|A|^2 = \hbar$. (The phase of A can be absorbed in the definition of the operators $a(\mathbf{p})$). Therefore the operator

$$\Phi(x) = \sqrt{\frac{\hbar}{4\pi}}\int \frac{d\mathbf{p}}{\sqrt{\omega(\mathbf{p})}}\Big(\,a(\mathbf{p})\,e^{ix\cdot p}\;+\;a^\dagger\,(\mathbf{p})\,e^{-ix\cdot p}\,\Big)\,. \tag{2.15}$$

is a canonical scalar field. It should be also noticed that

$$\Delta(x) = [\,\Phi(x),\ \Phi(0)\,] = 2i\int d\mathbf{p}\,|c(\mathbf{p})|^2\ \sin(x\cdot p)\,. \tag{2.16}$$

vanishes for causally disconnected points. The above field is therefore also causal, by the construction we adopt.

In conclusion, we found that, up to an irrelevant phase (that of A), Φ is univocally defined by eq.(1.1). This will not be true for the de Sitter group, as we now show.

3. The SO$(1,2)$ group

The starting point of our approach is the UIR of the $\mathrm{SO}_0(1,n)$ group, the group of linear transformations with determinant 1 which leaves $-(X^0)^2 + (X^1)^2 + \cdots + (X^n)^2$ invariant and which are connected to the identity. This is the isometry group of dS_n, the de Sitter space with n space-time dimensions. The UIR were first analyzed by Bargmann [22] for $n = 2$, Gelfand and Naimark for $n = 3$ [23], Thomas, Newton and Dixmier for $n = 4$ [24, 25]. General aspects for all n were studied in [26, 27, 28, 29]. In this paper we shall be concerned with the principal series representations.[2]

In the following we shall concentrate on the two-dimensional de Sitter space with group $\mathrm{SO}_0(1,2)$. Let $\mathcal{J}$ be the generator of the rotation subgroup and $\mathcal{K}_1$ and $\mathcal{K}_2$ the two boosts. They verify the commutation relations:

$$[\,\mathcal{J},\,\mathcal{K}_1\,] = i\mathcal{K}_2\,,\qquad [\,\mathcal{J},\,\mathcal{K}_2\,] = -i\mathcal{K}_1\,,\qquad [\,\mathcal{K}_1,\,\mathcal{K}_2\,] = -i\mathcal{J}\,. \tag{3.1}$$

[2]In a contraction limit [31], one can relate these representations of the dS group to those of the Poincaré group. All massive representations of the Poincaré group can be obtained this way [28]. However, the resulting representations are not irreducible, since one obtains a product of a positive and a negative energy UIR. This doubling is directly related to the afore mentioned two-parameter ambiguity in constructing local fields from UIR of the dS group.

The quadratic Casimir operator

$$\mathcal{C} = \mathcal{J}^2 - \mathcal{K}_1^2 - \mathcal{K}_2^2\,, \tag{3.2}$$

commutes with all the generators and is constant on an irreducible representation. Bargmann classified the UIR according to the value of $\mathcal{C}$ and the eigenvalues m of $\mathcal{J}$:

- (i) the principal series with $\mathcal{C} \leq -\frac{1}{4}$, $\quad m = 0, \pm 1, \ldots$ or $m = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$;
- (ii) the complementary series with $-\frac{1}{4} < \mathcal{C} < 0$ and $m = 0, \pm 1, \ldots$;
- (iii) the discrete series D_k^+ with k non-negative integer or half integer, $\mathcal{C} = k(k+1)$ and $m = k+1, k+2, \ldots$ and finally
- (iv) the discrete series D_k^-, $\mathcal{C} = k(k+1)$ and $m = -(k+1), -(k+2), \ldots$

For the representations of the principal series, the value of the Casimir operator is given by $-\left(\mu^2 + 1/4\right)$ with real μ. At fixed μ the generators of $\mathrm{SO}_0(1,2)$ act on the basis of the normalized eigenstate of $\mathcal{J}$ as

$$\mathcal{J}|m\rangle = m\,|m\rangle\,, \qquad \mathcal{K}_\pm|m\rangle = \left[\pm i\, m - i\left(i\mu - \frac{1}{2}\right)\right]|m \pm 1\rangle\,. \tag{3.3}$$

where the raising and lowering operators are defined as $\mathcal{K}_\pm = \mathcal{K}_1 \pm i\mathcal{K}_2$.

The representations with integer m can be also realized on the Hilbert space of square integrable complex (univalued) functions on the circle, $\Psi(\phi)$, equipped with the standard $\mathcal{L}^2$ scalar product. We shall use the Dirac notation where $\Psi(\phi) = \langle\phi|\Psi\rangle$.

The action of the group generators has the following expressions:

$$\begin{aligned}
\langle\phi|\mathcal{J}|\Psi\rangle &= -i\frac{d}{d\phi}\langle\phi|\Psi\rangle\,,\\
\langle\phi|\mathcal{K}_1|\Psi\rangle &= \left[i\sin\phi\frac{d}{d\phi} - i\left(i\mu - \frac{1}{2}\right)\cos\phi\right]\langle\phi|\Psi\rangle = \left[\frac{i}{2}\left\{\sin\phi, \frac{d}{d\phi}\right\} + \mu\cos\phi\right]\langle\phi|\Psi\rangle\,,\\
\langle\phi|\mathcal{K}_2|\Psi\rangle &= \left[-i\cos\phi\frac{d}{d\phi} - i\left(i\mu - \frac{1}{2}\right)\sin\phi\right]\langle\phi|\Psi\rangle = \left[-\frac{i}{2}\left\{\cos\phi, \frac{d}{d\phi}\right\} + \mu\sin\phi\right]\langle\phi|\Psi\rangle\,.
\end{aligned} \tag{3.4}$$

They are hermitian if μ is real.

It will be usefull to also have the action of finite transformations. They are given by

$$\begin{aligned}
\langle\phi|\, e^{i\theta\mathcal{J}}|\Psi\rangle &= \langle\phi + \theta|\Psi\rangle\,,\\
\langle\phi|\, e^{i\rho\mathcal{K}_1}|\Psi\rangle &= (\cosh\rho + \sinh\rho\cos\phi)^{i\mu - 1/2}\,\langle\phi_1|\Psi\rangle\,,\\
\langle\phi|\, e^{i\lambda\mathcal{K}_2}|\Psi\rangle &= (\cosh\lambda + \sinh\lambda\sin\phi)^{i\mu - 1/2}\,\langle\phi_2|\Psi\rangle\,.
\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}
\cos\phi_1 &= \frac{\cos\phi\cosh\rho + \sinh\rho}{\cosh\rho + \sinh\rho\cos\phi}\,, \qquad & \sin\phi_1 &= \frac{\sin\phi}{\cosh\rho + \sinh\rho\ \cos\phi}\,,\\
\cos\phi_2 &= \frac{\cos\phi}{\cosh\lambda + \sinh\lambda\ \sin\phi}\,, \qquad & \sin\phi_2 &= \frac{\sin\phi\cosh\lambda + \sinh\lambda}{\cosh\lambda + \sinh\lambda\ \sin\phi}\,.
\end{aligned} \tag{3.6}$$

4. Massive scalar field in de Sitter space

Let $|\Omega\rangle$ be a state carrying the trivial representation and $\mathscr{H}$ the Hilbert space carrying the UIR of the preceeding section. On the Fock space constructed from these two subspaces we define the creation and annihilation operators a_m and $a_m^\dagger$ as in eq.(2.4).

The (reducible) representation of the dS group on the Fock space is deduced from the irreducible representation by

$$U = \sum_{m,m'} \mathcal{U}_{mm'}\, a_m^\dagger\, a_{m'} \,, \tag{4.1}$$

where $\mathcal{U}_{mm'} = \langle m|\,\mathcal{U}\,|m'\rangle$. Each n-particle sector of the Fock space is thus kept invariant under the action of the group transformations.

Using these operators, we shall now construct a local field on dS space: $\Phi(x)$. We shall use the *global* coordinate system (t,θ), where t is the time coordinate and varies from $-\infty$ to $+\infty$ and θ is an angle coordinate. In this coordinate system, the metric is

$$ds^2 = -dt^2 + \cosh^2 t\, d\theta^2 \,. \tag{4.2}$$

The de Sitter space can be described by its embedding in a flat three dimensional space $-(X^0)^2 + (X^1)^2 + (X^2)^2 = 1$ with

$$\begin{aligned} X^0 &= \sinh t \,, \\ X^1 &= \cosh t \cos\theta \,, \\ X^2 &= \cosh t \sin\theta \,. \end{aligned} \tag{4.3}$$

The point with global coordinates $(0,0)$ is transported to (t,θ) by the following *ordered* sequence: first one acts with a boost generated by K_1 with parameter t to reach $(t,0)$, and then one reaches (t,θ) by a rotation with angle θ. Notice also that the point $(0,0)$ is left invariant by the boost generated by K_2.

As in eq.(2.6), the covariance condition (1.1) implies that $\Phi(t,\theta)$ can be deduced from $\Phi(0,0)$. Taking into account the non-commuting character of J and K_1 we have the central equation:

$$\Phi(t,\theta) = e^{-iJ\theta} e^{itK_1}\, \Phi(0,0)\, e^{-itK_1} e^{iJ\theta} \,. \tag{4.4}$$

We have adopted a sign convention such that the expanding phase of dS space corresponds to positive t. As a direct consequence we have

$$-i\partial_t \Phi(t,\theta) = [\, K_1 \cos\theta + K_2 \sin\theta,\ \Phi(t,\theta)\,] \,, \tag{4.5}$$

$$i\partial_\theta \Phi(t,\theta) = [\, J,\ \Phi(t,\theta)\,] \,. \tag{4.6}$$

As in Minkowski space, the field operator $\Phi(0,0)$ must be invariant under transformations that leave the point $(0,0)$ invariant. From the dS group only K_2 leaves $(0,0)$ invariant. Hence we have

$$[\, K_2,\ \Phi(0,0)\,] = 0 \,. \tag{4.7}$$

Using eq.(4.4) this condition leads to

$$[\, \cosh t\,(\cos\theta K_2 - \sin\theta K_1) + \sinh t J,\ \Phi(t,\theta)\,] = 0 \,. \tag{4.8}$$

Using eq.(4.5-4.8), we get

$$\begin{aligned}
[\,K_1,\,\Phi(t,\theta)\,] &= i\,(\tanh t\sin\theta\partial_\theta - \cos\theta\partial_t)\Phi(t,\theta)\,,\\
[\,K_2,\,\Phi(t,\theta)\,] &= -i\,(\tanh t\cos\theta\partial_\theta + \sin\theta\partial_t)\Phi(t,\theta)\,.
\end{aligned} \tag{4.9}$$

The right hand sides of the above equations coincide as they should with the Killing vector fields on dS space acting on the scalar field. The Casimir relation for the group generators gives rise to

$$\Big[J,[J,\Phi]\Big] - \Big[K_1,[K_1,\Phi]\Big] - \Big[K_2,[K_2,\Phi]\Big] = -\left(\mu^2+\frac{1}{4}\right)\Phi\,. \tag{4.10}$$

Upon using eq.(4.9) one verifies that $\Phi(t,\theta)$ obeys the Klein-Gordon equation:

$$\left(\partial_t^2 + \tanh t\,\partial_t - \frac{1}{\cosh^2 t}\,\partial_\theta^2\right)\Phi(t,\theta) = -\left(\mu^2+\frac{1}{4}\right)\Phi(t,\theta)\,. \tag{4.11}$$

This allows the identification of the mass squared in terms of μ: $M^2+\xi R = H^2(\mu^2+\frac{1}{4})$, where ξ is, in the action formalism, the coefficient of the term in $\frac{1}{2}R\Phi^2$.

We now search for solutions of eq.(4.7) which are linear in the creation and annihilation operators. This is our second assumption. We start in the Fourier basis and expand $\Phi(0,0)$ as

$$\Phi(0,0) = \sum_{m=-\infty}^{\infty} c_m a_m^\dagger + c_m^* a_m\,. \tag{4.12}$$

The field operator is thus fully determined by the c-number constants c_m. The covariance condition implies eq.(4.7), which in turn determines the constants c_m as we now show. From

$$[\,K_2,\,a_m^\dagger\,] = \frac{1}{2}\left[\left(m-\left(i\mu-\frac{1}{2}\right)\right)a_{m+1}^\dagger + \left(m+\left(i\mu-\frac{1}{2}\right)\right)a_{m-1}^\dagger\right]\,, \tag{4.13}$$

we get

$$c_{m-1}\left(m-\frac{1}{2}-i\mu\right) = -c_{m+1}\left(m+\frac{1}{2}+i\mu\right)\,. \tag{4.14}$$

This equation determines the constants c_m in terms of c_0 and c_1 for m even and odd respectively. More explicitly the above relation gives

$$\frac{c_{m+2}}{c_m} = \frac{\gamma_{m+2}}{\gamma_m}\,, \tag{4.15}$$

with

$$\gamma_m = e^{\frac{im\pi}{2}}\,\frac{\Gamma\left(\frac{m}{2}+\frac{1}{4}-\frac{i\mu}{2}\right)}{\Gamma\left(\frac{m}{2}+\frac{3}{4}+\frac{i\mu}{2}\right)}\,, \tag{4.16}$$

which implies

$$c_{2m} = \frac{c_0}{\gamma_0}\gamma_{2m}\,,\quad c_{2m+1} = \frac{c_1}{\gamma_1}\gamma_{2m+1}\,. \tag{4.17}$$

Notice that the coefficients c_m are even: $c_m = c_{-m}$, as for the Poincaré group. This guarantees the invariance under the parity transformation P. Parity acts on the UIR as $\mathsf{P}|m\rangle = |-m\rangle$ and on the field as $\mathsf{P}\,\Phi(t,\theta)\,\mathsf{P}^{-1} = \Phi(t,-\theta)$.

For de Sitter group it is also instructive to work out the arbitrariness in the expansion of $\Phi(0,0)$ in the position basis $|\phi\rangle$. We find convenient to present the analysis in terms of states in the UIR rather than field operators. We shall therefore study the states $|\Psi_0\rangle = \Phi(0,0)|\Omega\rangle$ which are invariant under K_2, namely they must obey $K_2|\Psi_0\rangle = 0$. Conversely an invariant state defines coefficients c_m obeying the above equations. In the ϕ representation the equation $\langle\phi|K_2|\Psi_0\rangle = 0$ gives rise to a singular first order equation:

$$\cos\phi \frac{d\Psi_0(\phi)}{d\phi} + \left(i\mu - \frac{1}{2}\right)\sin\phi\,\Psi_0(\phi) = 0\,. \tag{4.18}$$

Its solution is given by

$$\Psi_0(\phi) = \begin{cases} A\,(\cos\phi)^{i\mu-\frac{1}{2}}, & \text{for } -\frac{\pi}{2} < \phi < \frac{\pi}{2}\,, \\ B\,(-\cos\phi)^{i\mu-\frac{1}{2}}, & \text{for } \frac{\pi}{2} < \phi < \frac{3\pi}{2}\,. \end{cases} \tag{4.19}$$

It is important to notice that although the equation is first order, being singular at $\phi = \pm\frac{\pi}{2}$, its solution depends on two complex numbers. The region $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$ is the spatial region in the causal past of $(0,0)$: if a signal is emitted from this region at a sufficiently early time it can reach the space-time point (0,0) as can be seen in Figure 2.

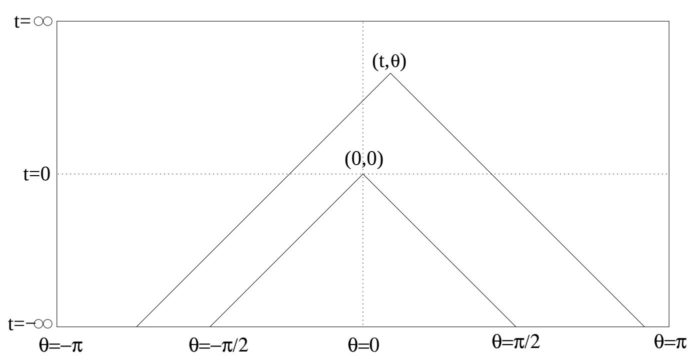

Figure 2: The Carter-Penrose diagram of the two dimensional dS space. We have represented the causal past of the point (t,θ).

As it will become clear in the sequel, there is another interesting way to express the general solution of eq.(4.18). This takes into account the cuts at the bifurcating horizons $\phi = \pm\frac{\pi}{2}$. Introducing the complex variable $z = \cos\phi$, the general solution can be written as

$$\Psi_0(z) = C\,(z - i\epsilon)^{i\mu-\frac{1}{2}} + D\,(z + i\epsilon)^{i\mu-\frac{1}{2}}\,, \tag{4.20}$$

where the limit $\epsilon \to 0^+$ is understood. The function $(z - i\epsilon)^{i\mu-\frac{1}{2}}$ is holomorphic in the lower half plane. The constants are related by

$$A = C + D\,, \qquad B = (e^{-i\pi})^{i\mu-\frac{1}{2}}\,C + (e^{i\pi})^{i\mu-\frac{1}{2}}\,D\,. \tag{4.21}$$

To exploit the arbitrariness of the A and B coefficients, we introduce the following operators:

$$a^\dagger(\phi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{im\phi} a_m^\dagger \,. \tag{4.22}$$

which create a position eigenstate when acting on the invariant vacuum: $a^\dagger(\phi)|\Omega\rangle = |\phi\rangle$. The field at the origin can then be written as

$$\Phi(0,0) = A \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} d\phi \, (\cos\phi)^{i\mu - \frac{1}{2}} a^\dagger(\phi) + B \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\phi \, (-\cos\phi)^{i\mu-\frac{1}{2}} a^\dagger(\phi) + \text{h.c.} \tag{4.23}$$

Let us define Φ_A, the creation part of the field operator with support inside the horizon, by

$$\Phi_A^+(0,0) = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} d\phi \, (\cos\phi)^{i\mu-\frac{1}{2}} a^\dagger(\phi) \,. \tag{4.24}$$

Then eq.(4.23) can be written as

$$\Phi(0,0) = A\, \Phi_A^+(0,0) + B\, e^{i\pi J} \Phi_A^+(0,0) e^{-i\pi J} + \text{h.c.} \tag{4.25}$$

Therefore, by direct computation, we get that the field operator $\Phi(x)$ is given, at any point $x = (t, \theta)$, by

$$\Phi(x) = A\, \Phi_A^+(x) + B\, \Phi_A^+(\bar{x}) + \text{h.c}, \tag{4.26}$$

where $\bar{x}$ is the antipodal point of coordinates $(-t, \pi + \theta)$. Notice that we have $\Phi_{A,B}(x) = \Phi_{B,A}(\bar{x})$.

In brief eq.(4.26) demonstrates that the general solution of the covariance condition can be written as a superposition of two local fields living on either side of the *transported* horizon as explained in Figure 2. At this point, it is interesting to notice that when $A = B$, the field operator is well defined on the orbifold $dS_2/\mathbb{Z}_2$, the elliptic de Sitter space [37] which has recently received some attention [38]. As we will show in the next Section, this choice is not compatible with the equal time commutation relations, in other words, the corresponding field operator is not canonical.

We can now relate the Fourier coefficients c_m to the above position representation. The constants c_0 and c_1, which fully determine the solution, can be directly related to the constants A and B. In fact, using the integral:

$$\int_0^{\pi/2} d\phi \, (\cos\phi)^\gamma = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1+\gamma}{2}\right)}{2\,\Gamma\left(\frac{\gamma}{2}+1\right)} \,, \tag{4.27}$$

we have,

$$c_0 = \frac{\gamma_0}{\sqrt{2}} \frac{\Gamma\left(i\frac{\mu}{2}+\frac{1}{4}\right)}{\Gamma\left(-i\frac{\mu}{2}+\frac{1}{4}\right)}(A+B), \qquad c_1 = -i\frac{\gamma_1}{\sqrt{2}} \frac{\Gamma\left(i\frac{\mu}{2}+\frac{3}{4}\right)}{\Gamma\left(-i\frac{\mu}{2}+\frac{3}{4}\right)}(A-B) \,. \tag{4.28}$$

4.1 Commutation relations

In the preceding section, we showed that the covariance properties determine the field operator up to two complex constants c_0 and c_1, or equivalently A and B. Here, we calculate the equal time commutators between two fields and between the field and its time derivative. It turns out that the first commutator vanishes for all constants and that the second is proportional to a delta function, as in the case of the Poincaré group. Therefore any field, solution of eq.(4.4), is canonical. Similarly to what was done is Section 2, it is then possible to find the identification of $\hbar$. The novelty is that the field is no longer unique. We shall show indeed that the moduli space of solutions is non trivial since it is given by $\mathrm{SU}(1,1)/\mathrm{U}(1)$. In addition, the two commutators and the Klein-Gordon equation imply that the canonical fields are causal.

Consider first

$$g(t,\theta) = [\,\Phi(t,0),\ \Phi(t,\theta)\,]\,. \tag{4.29}$$

It is a c-number. Multiplying this expression on the right by $e^{-i\theta J}$ and on the left by $e^{i\theta J}$ we get

$$\begin{aligned} g(t,\theta) &= e^{-i\theta J}\, g(t,\theta)\, e^{i\theta J} \\ &= [\,\Phi(t,-\theta),\ \Phi(t,0)\,] \\ &= -g(t,-\theta)\,. \end{aligned} \tag{4.30}$$

On the other hand, multiplying by the parity operator P we have

$$\begin{aligned} g(t,\theta) &= \mathsf{P}\, g(t,\theta)\, \mathsf{P}^{-1} \\ &= [\,\Phi(t,0),\ \Phi(t,-\theta)\,] \\ &= g(t,-\theta)\,. \end{aligned} \tag{4.31}$$

From these two equations we get

$$[\,\Phi(t,0),\ \Phi(t,\theta)\,] = 0\,. \tag{4.32}$$

The field operators at equal time thus commute for all choices of the constants c_0 and c_1.

Consider next

$$h(\theta) = [\,\partial_t\Phi(0,0),\ \Phi(0,\theta)\,]\,. \tag{4.33}$$

We have the following equalities

$$\begin{aligned} h(\theta) &= i\Big[\,[\,K_1,\ \Phi(0,0)\,],\ \Phi(0,\theta)\Big] \\ &= i\Big[\,e^{i\theta J}[\,K_1,\ \Phi(0,0)\,]e^{-i\theta J},\ \Phi(0,0)\Big] \\ &= i\Big[\,[\,\cos\theta K_1 - \sin\theta K_2,\ \Phi(0,-\theta)\,],\ \Phi(0,0)\Big]\,, \end{aligned} \tag{4.34}$$

where we have used the definition of $\Phi(t,\theta)$ in terms of $\Phi(0,0)$, the c-number character of $h(\theta)$ and the transformation of K_1 under a rotation. Now use the Jacobi identity as well as the previously proved identity $g(0,\theta) = 0$ and $[K_2, \Phi(0,0)] = 0$ to obtain

$$\begin{aligned} h(\theta) &= i\Big[\,[\,\cos\theta K_1,\ \Phi(0,0)\,],\ \Phi(0,-\theta)\Big] \\ &= \cos\theta\, h(-\theta)\,. \end{aligned} \tag{4.35}$$

On the other hand, we have

$$
\begin{aligned}
h(\theta) &= i\Big[\,\mathsf{P}[\,K_1,\,\Phi(0,0)\,]\mathsf{P}^{-1},\,\mathsf{P}\Phi(0,\theta)\mathsf{P}^{-1}\Big] \\
&= i\Big[\,[\,K_1,\,\Phi(0,0)\,],\,\Phi(0,-\theta)\Big] \\
&= h(-\theta)\,,
\end{aligned}
\tag{4.36}
$$

where we used $\mathsf{P}K_1\mathsf{P}^{-1} = K_1$. Finally, using eq.(4.35) and eq.(4.36), we obtain

$$
(1-\cos\theta)\,h(\theta) = 0\,,
\tag{4.37}
$$

with $h(\theta)$ even in θ. Since we work on the circle, the solution is

$$
h(\theta) = iN\,\delta(\theta)\,.
\tag{4.38}
$$

The constant N is real since $\Phi(t,\theta)$ is hermitian. Therefore, unless N is equal to zero the covariant field Φ is canonical.

We now determine N in terms of c_0 and c_1. We need

$$
[\,K_1,\,a^\dagger_m\,] = \frac{i}{2}\left[\left(m - i\mu + \frac{1}{2}\right)a^\dagger_{m+1} - \left(m + i\mu - \frac{1}{2}\right)a^\dagger_{m-1}\right],
\tag{4.39}
$$

in order to calculate $[\,K_1,\,\Phi(0,0)\,]$. Since

$$
\int_0^{2\pi}\frac{d\theta}{2\pi}\,\Phi(0,\theta) = c_0\,a^\dagger_0 + c^*_0\,a_0\,,
\tag{4.40}
$$

and $c_{-1} = c_1$ we get

$$
\frac{N}{2\pi} = -2\,\mathrm{Re}\left[\left(\mu - \frac{i}{2}\right)c^*_0 c_1\right].
\tag{4.41}
$$

The Planck constant $\hbar$ is introduced as $N = -\hbar$. In the following it will be set to one and thus

$$
\mathrm{Re}\left[\left(\mu - \frac{i}{2}\right)c^*_0 c_1\right] = \frac{1}{4\pi}\,.
\tag{4.42}
$$

Defining the 2-component vector:

$$
\mathbf{z} = \frac{c_0}{\gamma}\sqrt{2\pi}\begin{pmatrix}1\\1\end{pmatrix} + \gamma^* c_1\left(\mu - \frac{i}{2}\right)\sqrt{2\pi}\begin{pmatrix}1\\-1\end{pmatrix},
\tag{4.43}
$$

where γ is an arbitrary complex constant, eq.(4.42) reads

$$
\mathbf{z}^\dagger\sigma_{\mathbf{3}}\mathbf{z} = 1\,,
\tag{4.44}
$$

where $\sigma_{\mathbf{3}} = diag(1,-1)$ is the third Pauli matrix. The invariance group of this equation is U$(1,1)$. Since a fixed two component vector is invariant under a subgroup U(1), the solutions of eq.(4.42) parameterize the coset U$(1,1)$/U(1). In addition, the overall phase in $\mathbf{z}$ has no physical meaning, the moduli space is thus SU$(1,1)$/U(1). We notice that this is also the moduli space of the alpha vacua [11]. This connection will be made more precise in the next section. A convenient parameterization of $\mathbf{z}$ is given by

$$
\mathbf{z} = \begin{pmatrix}\cosh\alpha\\ e^{i\beta}\sinh\alpha\end{pmatrix},
\tag{4.45}
$$

where the two components will be shown to be Bogoliubov coefficients when interpreted in the usual field theoretical description.

4.2 Time reversal

One gains a better understanding of the field properties when considering the discrete transformations which leave the point $(0,0)$ invaraint.

As we have already seen, invariance under parity $\phi \to -\phi$ is automatically realized since $c_{-m} = c_m$ for all choices of c_0 and c_1.

Invariance under time reversal is more instructive. It is implimented by $\mathsf{T}e^{i\theta J}\mathsf{T}^{-1} = e^{i\theta J}$ and $\mathsf{T}e^{i\omega^a K_a}\mathsf{T}^{-1} = e^{-i\omega^a K_a}$. It can be realized in two different ways [27]. The first one is by a unitary operator as

$$\mathsf{T}|m\rangle = \epsilon(-1)^m|m\rangle\,, \tag{4.46}$$

with $\epsilon = \pm 1$. The unitary time inversion operator acts on the field as

$$\mathsf{T}\Phi(0,0)\mathsf{T}^{-1} = \sum_{m=-\infty}^{\infty} (-1)^m c_m\, a_m^\dagger + (-1)^m c_m^*\, a_m\,, \tag{4.47}$$

that is $\mathsf{T}\Phi(t,\theta)\mathsf{T}^{-1} = \Phi(-t,\theta+\pi)$. If we demand that $\mathsf{T}\Phi(0,0)\mathsf{T}^{-1} = \Phi(0,0)$ then we get that $c_{2m+1} = 0$ for $\epsilon = 1$ or $c_{2m} = 0$ for $\epsilon = -1$. These two relations lead to a vanishing N and so are incompatible with the requirement of the canonical commutation relations, so we must use the second way of realizing the time reversal invariance.

The second one is, as in flat space-time, by an anti-unitary operator. In this case, T acts on the basis $|m\rangle$ as

$$\mathsf{T}|m\rangle = e^{i\nu_m}|-m\rangle\,, \tag{4.48}$$

with

$$e^{i(\nu_{m+1}-\nu_m)} = -\frac{m+\frac{1}{2}-i\mu}{m+\frac{1}{2}+i\mu} = \frac{\gamma_{m+1}}{\gamma_{m+1}^*}\frac{\gamma_m^*}{\gamma_m}\,. \tag{4.49}$$

The last relation implies that

$$e^{i\nu_m} = e^{i\delta}\frac{\gamma_m}{\gamma_m^*}\,, \tag{4.50}$$

where δ is common unobservable phase which can be put to zero without loss of generality. When acting on the field we get

$$\mathsf{T}\Phi(0,0)\mathsf{T}^{-1} = \sum_{m=-\infty}^{\infty} c_m^* e^{i\nu_m}\, a_m^\dagger + c_m e^{-i\nu_m}\, a_m\,, \tag{4.51}$$

That is the field $\mathsf{T}\Phi(0,\theta)\mathsf{T}^{-1}$ is of the same form as $\Phi(0,\theta)$ with the coefficients c_m replaced by $c_m^* e^{i\nu_m}$:

$$\mathsf{T}\Phi(t,\theta)_{\{c_m\}}\mathsf{T}^{-1} = \Phi(-t,\theta)_{\{c_m^* e^{i\nu_m}\}}\,. \tag{4.52}$$

Notice that $c_m^* e^{i\nu_m}$ verify the defining relations of the c_m as they should.

If we require $\mathsf{T}\Phi(0,0)\mathsf{T}^{-1} = \Phi(0,0)$ then we get

$$c_m^* e^{i\nu_m} = c_m\,. \tag{4.53}$$

This implies that the phase of c_m is $\nu_m/2$. This condition is compatible with eq.(4.14), and also fixes the physically meaningful relative phase between c_0 and c_1. Explicitely, the anti-unitary time inversion invariance gives

$$\frac{c_1 c_0^*}{c_1^* c_0} = -\frac{\frac{1}{2} - i\mu}{\frac{1}{2} + i\mu} \, . \tag{4.54}$$

This condition can be written as

$$\mathrm{Im}\left[\left(\mu - \frac{i}{2}\right) c_0^* c_1\right] = 0 \, . \tag{4.55}$$

The canonical commutation relation and time reversal can thus be regrouped in a single simple formula:

$$\left(\mu - \frac{i}{2}\right) c_0^* c_1 = \frac{1}{4\pi} \, . \tag{4.56}$$

In terms of the parameterization of eq.(4.45), we simply get $\beta = 0$.

5. Two point function and Bunch-Davies vacuum

As briefly explained in the Introduction, the easiest way to make contact with the usual treatment of quantum fields on dS space is through computation of the two point-function which is an observable.

In the present approach, the two point function evaluated in the unique invariant vacuum state is given by

$$\langle \Omega | \Phi(0,0) \Phi(0,\theta) | \Omega \rangle = \sum_{m=-\infty}^{\infty} |c_m|^2 e^{-im\theta} \, . \tag{5.1}$$

Imposing the Hadamard form of this two point function, that is, the same behavior in the coincidence point limit as that of the flat case, gives a constraint on the asymptotic value of the c_m. Recall that the flat two point function on the circle is given by

$$\frac{1}{4\pi} \sum_{m=-\infty}^{\infty} \frac{1}{\sqrt{m^2 + M^2}} e^{-im\theta} \, . \tag{5.2}$$

At short distances, it reduces to

$$-\frac{1}{2\pi} \log \theta \, . \tag{5.3}$$

The asymptotic behavior of c_m can be determined from eq.(4.16). One finds

$$|c_{2m}|^2 = \frac{|c_0|^2}{|\gamma_0|^2} |\gamma_{2m}|^2 \underset{m \to \infty}{\approx} \frac{|c_0|^2}{|\gamma_0|^2} \frac{1}{|m|} \, , \qquad |c_{2m+1}|^2 = \frac{|c_1|^2}{|\gamma_1|^2} |\gamma_{2m+1}|^2 \underset{m \to \infty}{\approx} \frac{|c_1|^2}{|\gamma_1|^2} \frac{1}{|m|} \, . \tag{5.4}$$

This has the dependence on m required so that the two point function coincides with the flat one at short distances. Indeed we get

$$-2 \left(\left| \frac{c_0}{\gamma_0} \right|^2 + \left| \frac{c_1}{\gamma_1} \right|^2 \right) \log \theta \, . \tag{5.5}$$

Comparison of the overall constants gives

$$\left|\frac{c_0}{\gamma_0}\right|^2 + \left|\frac{c_1}{\gamma_1}\right|^2 = \frac{1}{4\pi}\,. \tag{5.6}$$

From this equation and the canonical commutation relation which reads

$$\mathrm{Re}\left[\left(\frac{c_0}{\gamma_0}\right)^* \left(\frac{c_1}{\gamma_1}\right)\right] = \frac{1}{8\pi}\,, \tag{5.7}$$

we get

$$\left|\frac{c_0}{\gamma_0} - \frac{c_1}{\gamma_1}\right|^2 = 0\,. \tag{5.8}$$

This gives (up to a choice of the unphysical phase)

$$c_0^{\mathrm{BD}} = \frac{1}{\sqrt{8\pi}}\,\gamma_0\,, \qquad c_1^{\mathrm{BD}} = \frac{1}{\sqrt{8\pi}}\,\gamma_1\,. \tag{5.9}$$

Therefore the field operator is characterized for all m by

$$c_m^{\mathrm{BD}} = \frac{1}{\sqrt{8\pi}}\,\gamma_m\,. \tag{5.10}$$

We have called this solution Bunch-Davies (BD) because it possesses a fixed sign of the conformal frequency in the large m limit, see next section for further discussion. In the limit of large μ we obtain

$$c_0^{\mathrm{BD}},\, c_1^{\mathrm{BD}} \underset{\mu\to\infty}{\approx} \frac{1}{\sqrt{4\pi\mu}}\,, \tag{5.11}$$

which is expected from flat space. Notice that the BD vacuum is time reversal invariant, in the sense of eq.(4.53).

In terms of (A, B) and (C, D), the BD field operator is characterized, up to an overall phase, by

$$\begin{aligned} A^{\mathrm{BD}} &= \frac{e^{\frac{\pi\mu}{2}}}{\sqrt{8\pi\cosh\pi\mu}}\,,\; B^{\mathrm{BD}} = \frac{-ie^{-\frac{\pi\mu}{2}}}{\sqrt{8\pi\cosh\pi\mu}} = -ie^{-\pi\mu}A^{\mathrm{BD}}\,,\\ C^{\mathrm{BD}} &= 0\,, \qquad\qquad D^{\mathrm{BD}} = \frac{e^{\frac{\pi\mu}{2}}}{\sqrt{8\pi\cosh\pi\mu}}\,. \end{aligned} \tag{5.12}$$

From the last two equations, we learn that D (C) is the amplitude of positive (negative) conformal frequency modes.

Using the BD solution as the reference solution in eq.(4.43), which amounts to choose $\gamma = \gamma_0$, the moduli space of canonical fields, eq.(4.45), can be parameterized by

$$\begin{aligned} &c_0 = c_0^{\mathrm{BD}}(\cosh\alpha + e^{i\beta}\sinh\alpha)\,, \qquad c_1 = c_1^{\mathrm{BD}}(\cosh\alpha - e^{i\beta}\sinh\alpha)\,,\\ &A = \cosh\alpha\, A^{\mathrm{BD}} + e^{i\beta}\sinh\alpha\, B^{\mathrm{BD}}\,,\; B = \cosh\alpha\, B^{\mathrm{BD}} + e^{i\beta}\sinh\alpha\, A^{\mathrm{BD}}\,,\\ &C = -i\,e^{i\beta}\sinh\alpha\, e^{-\pi\mu}\, D^{\mathrm{BD}}\,, \qquad D = \cosh\alpha\, D^{\mathrm{BD}}. \end{aligned} \tag{5.13}$$

Using eq.(4.26) and defining the operator:

$$\Phi^+_{\mathrm{BD}}(x) = A^{\mathrm{BD}}\,\left(\Phi^+_A(x) - ie^{-\pi\mu}\Phi^+_A(\bar{x})\right)\,, \tag{5.14}$$

the general field can be expressed as

$$\Phi_{\alpha,\beta}(x) = \cosh\alpha\, \Phi^{+}_{\rm BD}(x) + e^{i\beta} \sinh\alpha\, \Phi^{+}_{\rm BD}(\bar{x}) + \text{h.c.} \tag{5.15}$$

Let us denote the BD positive Wightman function by

$$G_0(x;y) = \langle\Omega|\, \Phi_{\rm BD}(x)\, \Phi_{\rm BD}(y)\, |\Omega\rangle\,. \tag{5.16}$$

It satisfies $G_0(y;x) = G_0^*(x;y)$ and $G_0(\bar{x};\bar{y}) = G_0(y;x)$. Using eq.(5.15), we can express the positive Wightman function of any field operator in terms of $G_0(x;y)$:

$$\begin{aligned} G_{\alpha,\beta}(x;y) &= \langle\Omega|\, \Phi_{\alpha,\beta}(x)\, \Phi_{\alpha,\beta}(y)\, |\Omega\rangle \\ &= \cosh^2\alpha\, G_0(x;y) \,+\, \sinh^2\alpha\, G_0(\bar{x};\bar{y}) \\ &\quad + \sinh\alpha\cosh\alpha \left(e^{i\beta}\, G_0(x;\bar{y}) \,+\, e^{-i\beta}\, G_0(\bar{x};y) \right)\,. \end{aligned} \tag{5.17}$$

This expression is identical to eq.(2.15) in [11] which gives the relation between the Wightman function evaluated in the alpha vacuum characterized by α and β, and that evaluated in the Bunch-Davies vacuum. The relationship between our approach and QFT in dS space will be further analyzed in section 7.

To conclude we notice that, in the present formalism, we can also consider the Wightman function between two different field operators, namely

$$\begin{aligned} \langle\Omega|\, \Phi_{\alpha,\beta}(x)\, \Phi_{\alpha',\beta'}(y)\, |\Omega\rangle &= \cosh\alpha\cosh\alpha'\, G_0(x;y) \,+\, e^{-i(\beta-\beta')} \sinh\alpha\sinh\alpha'\, G_0(\bar{x};\bar{y}) \\ &\quad + e^{i\beta'} \cosh\alpha\sinh\alpha'\, G_0(x;\bar{y}) \,+\, e^{-i\beta} \cosh\alpha'\sinh\alpha\, G_0(\bar{x};y)\,. \end{aligned} \tag{5.18}$$

It is then a legitimate question to ask in which circumstances this type of expression could occur in QFT. To get it, one should consider the cross term of the response function matrix associated with two local quantum systems coupled to two different fields. More explicitely, one should consider interaction Hamiltonians of the form (in the interacting picture):

$$H_{int} = g_1\, q_1(t)\, \Phi_{\alpha,\beta}(t,\mathbf{x}) + g_2\, q_2(t)\, \Phi_{\alpha',\beta'}(t,\mathbf{y})\,, \tag{5.19}$$

where q_1, q_2 are coordinates of the two systems and where g_1 and g_2 are the coupling constants. Then, to first order in $g_1 g_2$, the amplitude to induce transitions to both oscillators will be governed by eq.(5.18).

6. *in* and *out* vacuum

The explicit transformation of the state vector in the UIR recalled in Section 2 can be used to determine the full space-time dependence of the field operator, and this without using the Klein-Gordon equation. One can then use the asymptotic behavior of the field operator to identify the *in* and *out* positive frequency solutions.

In order to determine $\Phi(t,0)$ let us define the vector state $|\Psi_t\rangle = \Phi(t,0)|\Omega\rangle$. It is the boosted state $e^{itK_1}|\Psi_0\rangle$. In the ϕ-representation, using eq.(3.5), we get that $\Psi_t(\phi) = \langle\phi|\Psi_t\rangle$ is

$$\begin{aligned} \Psi_t(\phi) &= A\, \Theta(\cos\phi + \tanh t)\, (\cos\phi\cosh t + \sinh t)^{i\mu-\frac{1}{2}} \\ &\quad + B\, \Theta(-\cos\phi - \tanh t)\, (-\cos\phi\cosh t - \sinh t)^{i\mu-\frac{1}{2}}\,. \end{aligned} \tag{6.1}$$

The support of the term proportionnal to A is the space region in the causal past of $(t,0)$ (see Figure 2). From eq. (6.1), the field $\Phi(t,0)$ is

$$\begin{aligned}
\Phi(t,0) &= \int_0^{2\pi} d\phi\, \Psi_t(\phi)\, a^\dagger(\phi) + \Psi_t^*(\phi)\, a(\phi) \\
&= A \int_{-\pi+\arccos(\tanh t)}^{\pi-\arccos(\tanh t)} d\phi\, (\cosh t\cos\phi + \sinh t)^{i\mu-\frac{1}{2}}\ a^\dagger(\phi) + \text{h.c.} \\
&\quad + B \int_{\pi-\arccos(\tanh t)}^{\pi+\arccos(\tanh t)} d\phi\, (-\cosh t\cos\phi - \sinh t)^{i\mu-\frac{1}{2}}\ a^\dagger(\phi) + \text{h.c.}
\end{aligned} \tag{6.2}$$

Performing a rotation of angle θ we obtain

$$\begin{aligned}
\Phi(t,\theta) &= A \int_{-\pi+\arccos(\tanh t)}^{\pi-\arccos(\tanh t)} d\phi\, (\cosh t\cos\phi + \sinh t)^{i\mu-\frac{1}{2}}\ a^\dagger(\phi+\theta) + \text{h.c.} \\
&\quad + B \int_{\pi-\arccos(\tanh t)}^{\pi+\arccos(\tanh t)} d\phi\, (-\cosh t\cos\phi - \sinh t)^{i\mu-\frac{1}{2}}\ a^\dagger(\phi+\theta) + \text{h.c.}
\end{aligned} \tag{6.3}$$

To determine the field operators propagating positive frequencies in the infinite proper past and future it will be convenient to decompose the field in Fourier modes:

$$\Phi(t,\theta) = \sum_{m=-\infty}^{\infty} c_m(t)\ e^{-im\theta} a_m^\dagger + c_m^*(t)\ e^{im\theta} a_m\,. \tag{6.4}$$

We get

$$\begin{aligned}
c_m(t) &= A \int_{-\pi+\arccos(\tanh t)}^{\pi-\arccos(\tanh t)} \frac{d\phi}{\sqrt{2\pi}} (\cosh t\cos\phi + \sinh t)^{i\mu-\frac{1}{2}} e^{-im\phi} \\
&\quad + B \int_{\pi-\arccos(\tanh t)}^{\pi+\arccos(\tanh t)} \frac{d\phi}{\sqrt{2\pi}} (-\cosh t\cos\phi - \sinh t)^{i\mu-\frac{1}{2}} e^{-im\phi}\,.
\end{aligned} \tag{6.5}$$

In terms of the conformal time η defined by $e^t = \tan\frac{\eta}{2}$, we have

$$\begin{aligned}
c_m(\eta) &= A\ (\sin\eta)^{-i\mu+\frac{1}{2}} \int_{-\eta}^{\eta} \frac{d\phi}{\sqrt{2\pi}} (\cos\phi - \cos\eta)^{i\mu-\frac{1}{2}} e^{-im\phi} \\
&\quad + B\ (\sin\eta)^{-i\mu+\frac{1}{2}} \int_{\eta}^{2\pi-\eta} \frac{d\phi}{\sqrt{2\pi}} (-\cos\phi + \cos\eta)^{i\mu-\frac{1}{2}} e^{-im\phi}\,.
\end{aligned} \tag{6.6}$$

The integrals are "Mehler-Dirichlet" integral representations of the associated Legendre functions [39]. Explicitely, one has

$$c_m(\eta) = (\sin\eta)^{\frac{1}{2}}\,\Gamma\left(i\mu + \frac{1}{2}\right)\left(A\,\mathrm{P}^{-i\mu}_{-m-\frac{1}{2}}(\cos\eta) + (-1)^m B\,\mathrm{P}^{-i\mu}_{-m-\frac{1}{2}}(-\cos\eta)\right). \tag{6.7}$$

The limits $t\to\pm\infty$ correspond to $z=\cos\eta\to\mp1$. In these limits, the asymptotic behavior of the associated Legendre function is given by (see page 164 in [39])

$$\begin{aligned}
\mathrm{P}^{-i\mu}_{m-\frac{1}{2}}(z) &\underset{z\to1}{\approx} \frac{2^{-\frac{i\mu}{2}}}{\Gamma(1+i\mu)}(1-z)^{\frac{i\mu}{2}}\,, \\
\mathrm{P}^{-i\mu}_{m-\frac{1}{2}}(z) &\underset{z\to-1}{\approx} \frac{2^{\frac{i\mu}{2}}\Gamma(i\mu)}{\Gamma\left(m+\frac{1}{2}+i\mu\right)\Gamma\left(-m+\frac{1}{2}+i\mu\right)}(1+z)^{-\frac{i\mu}{2}} + (-1)^m\frac{2^{-\frac{i\mu}{2}}\Gamma(-i\mu)}{\pi}(1+z)^{\frac{i\mu}{2}}\,.
\end{aligned} \tag{6.8}$$

The asymptotic behavior of $c_m(t)$ in the early past can thus be determined as

$$
\begin{aligned}
c_m(t) &\underset{t\to-\infty}{\approx} \sqrt{2}\, e^{\frac{t}{2}}\, \Gamma\left(i\mu+\frac{1}{2}\right) \times \\
&\times \left[A\, \frac{1}{\Gamma(1+i\mu)} e^{i\mu t} + B \left(\frac{(-1)^m \Gamma(i\mu)}{\Gamma\left(m+\frac{1}{2}+i\mu\right)\Gamma\left(-m+\frac{1}{2}+i\mu\right)} e^{-i\mu t} + \frac{\Gamma(-i\mu)}{\pi} e^{i\mu t} \right) \right].
\end{aligned}
\tag{6.9}
$$

The solution with positive proper time frequency in the remote past thus corresponds to

$$
B^{\mathrm{IN}} = 0\,. \tag{6.10}
$$

This characterizes the *in* vacuum [10].

The asymptotic behavior of $c_m(t)$ in the late future can similarly be deduced from

$$
\begin{aligned}
c_m(t) &\underset{t\to\infty}{\approx} \sqrt{2}\, e^{-\frac{t}{2}}\, \Gamma\left(i\mu+\frac{1}{2}\right) \times \\
&\times \left[A \left(\frac{\Gamma(i\mu)}{\Gamma\left(m+\frac{1}{2}+i\mu\right)\Gamma\left(-m+\frac{1}{2}+i\mu\right)} e^{i\mu t} + \frac{(-1)^m}{\pi}\Gamma(-i\mu) e^{-i\mu t} \right) + B\, \frac{(-1)^m}{\Gamma(1+i\mu)} e^{-i\mu t} \right].
\end{aligned}
\tag{6.11}
$$

The *out* vacuum is thus characterized by

$$
A^{\mathrm{OUT}} = \frac{-\pi\, B^{\mathrm{OUT}}}{\Gamma(-i\mu)\Gamma(1+i\mu)} = i\sinh\pi\mu\, B^{\mathrm{OUT}}\,. \tag{6.12}
$$

It is easy to obtain the *in* to *out* transformation when noticing that *in* and *out* vacua are time-reversal states, which implies $\alpha^{\mathrm{IN}} = \alpha^{\mathrm{OUT}}$ and $\beta^{\mathrm{IN}} = -\beta^{\mathrm{OUT}}$, or $\mathbf{z}^{\mathrm{OUT}} = \mathbf{z}^{\mathrm{IN}*}$. Therefore, using the fact that $\beta^{\mathrm{IN}} = \frac{\pi}{2}$ and $\tanh\alpha^{\mathrm{IN}} = e^{-\pi\mu}$, the mean number of *out* quanta of momentum m in *in* vacuum is

$$
\begin{aligned}
\bar{n}_{\mathrm{OUT}} &= |\, e^{i\beta^{\mathrm{IN}}} \sinh\alpha^{\mathrm{IN}} \cosh\alpha^{\mathrm{IN}} + \text{c.c.}\,|^2 \\
&= \frac{1}{\sinh^2\pi\mu}\,,
\end{aligned}
\tag{6.13}
$$

which coincide with the 4-dimensional result, see eq.(32) in [10].

The large m behavior of $c_m(\eta)$ can also be determined as

$$
c_m(\eta) \underset{m\to\infty}{\approx} \frac{e^{-i\frac{\pi}{4}} e^{\frac{\mu\pi}{2}}}{\sqrt{2\pi}\, m!} \Gamma\left(m+\frac{1}{2}-i\mu\right)\Gamma\left(i\mu+\frac{1}{2}\right)\left[e^{im\eta}(A+ie^{-\mu\pi}B) + e^{-im\eta}(B+ie^{-\mu\pi}A)\right]. \tag{6.14}
$$

This clearly exhibits the defining property of the Bunch-Davies vacuum as having only positive conformal frequency for large momenta. Notice that this equation can also be expressed in terms of the coefficients C and D as

$$
c_m(\eta) \underset{m\to\infty}{\approx} \frac{e^{-i\frac{\pi}{4}} e^{\frac{\mu\pi}{2}}}{\sqrt{2\pi}\, m!} \Gamma\left(m+\frac{1}{2}-i\mu\right)\Gamma\left(i\mu+\frac{1}{2}\right)\left[e^{im\eta}(1+e^{-2\pi\mu})\, D + 2ie^{-im\eta}\cosh\pi\mu\, C\right]. \tag{6.15}
$$

This shows the holomorphic properties of the Bunch-Davies vacuum: $C = 0$.

7. Relation to field theory approach

The aim of this section is the following. Starting with the usual treatment of QFT on dS space, we construct a squeezing operator which transforms an operator annihilating an alpha vacuum into an operator annihilating another vacuum. When acting on the first alpha vacuum this operator engenders the second vacuum. We shall show that when it acts on the field operator corresponding to the first vacuum, one obtains the canonical field associated with the second one.

In field theory, the free scalar field is first decomposed into Fourier modes

$$\Phi(t,\theta) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} a_m\, u_m(t)\, e^{im\theta} \;+\; a_m^\dagger\, u_m^*(t)\, e^{-im\theta}\,, \tag{7.1}$$

where the time dependent modes satisfy the differential equation

$$\left(\frac{d^2}{dt^2} + \tanh t \frac{d}{dt} + \frac{m^2}{\cosh^2 t}\right) u_m(t) = -\left(\mu^2 + \frac{1}{4}\right) u_m(t)\,, \tag{7.2}$$

and the Wronskian condition

$$\cosh t \Big(u_m^*(t)\, \dot{u}_m(t) \;-\; \dot{u}_m^*(t)\, u_m(t) \Big) = -i\,, \tag{7.3}$$

so that the equal time canonical commutation relations are satisfied. When the boundary condition fixing u_m is parity invariant, the situation we shall consider, one also has

$$u_m(t) = u_{-m}(t)\,. \tag{7.4}$$

The solution u_m is not uniquely defined by the two requirements (7.2) and (7.3). In fact

$$\tilde{u}_m \;=\; e^{i\sigma_m} \cosh\rho_m\, u_m \;+\; e^{-i\lambda_m} \sinh\rho_m\, u^*_{-m}\,, \tag{7.5}$$

also satisfies the two requirement, where ρ_m, σ_m and λ_m are arbitrary real constants. Fixing the unphysical phase of $\tilde{u}_m$, we take herefrom σ_m to zero. The corresponding Bogoliubov transformation is given by

$$\tilde{a}_m \;=\; \cosh\rho_m\, a_m \;-\; e^{i\lambda_m} \sinh\rho_m\, a^\dagger_{-m}\,. \tag{7.6}$$

This transformation can be obtained by introducing the so-called squeezing operators

$$\mathcal{S}_{\rho_m,\lambda_m} = \begin{cases} \exp\left[\frac{\rho_0}{2} \left\{ e^{i\lambda_0}\, (a_0^\dagger)^2 \;-\; e^{-i\lambda_0}\, (a_0)^2 \right\} \right], & \text{for} \quad m=0\,, \\[2ex] \exp\left[\rho_m \left\{ e^{i\lambda_m}\, a_m^\dagger a_{-m}^\dagger \;-\; e^{-i\lambda_m}\, a_m a_{-m} \right\} \right], & \text{for} \quad m \neq 0\,. \end{cases} \tag{7.7}$$

We then have (see e.g. [40])

$$\tilde{a}_m \;=\; \mathcal{S}_{\rho_m,\lambda_m} a_m\, \mathcal{S}^{-1}_{\rho_m,\lambda_m}\,. \tag{7.8}$$

The relation between the two vacua is formally given by the product over all m:

$$|\Omega_{\{\rho_m,\lambda_m\}}\rangle \;=\; \left(\prod_{m=0}^{\infty} \mathcal{S}_{\rho_m,\lambda_m} \right) |\Omega\rangle\,. \tag{7.9}$$

The reason we wrote *formally* is the following. The squeezing operator $\mathcal{S}_{\rho_m,\lambda_m}$ possesses the normal ordered form:

$$\mathcal{S}_{\rho_m,\lambda_m} = \exp\Big[\, e^{i\lambda_m}\tanh\rho_m\, a_m^\dagger a_{-m}^\dagger \Big] \left(\frac{1}{\cosh\rho_m}\right)^{1+a_m^\dagger a_m + a_{-m}^\dagger a_{-m}} \exp\Big[-e^{-i\lambda_m}\tanh\rho_m\, a_m a_{-m}\Big]\,. \tag{7.10}$$

The product over m of the squeezing operators contains an overall factor given by

$$\prod_{m=1}^{\infty}\frac{1}{\cosh\rho_m}\,. \tag{7.11}$$

The product is a well defined unitary operator provided the above coefficient is nonvanishing, which implies $\rho_m^2 \to 0$ faster than $1/m$.

In general the vacuum $|\Omega_{\{\rho_m,\lambda_m\}}\rangle$ is not dS invariant, that is, the Wightman function in this vacuum:

$$\begin{aligned} G_{\{\rho_m,\lambda_m\}}(t,\theta;t',\theta') &= \langle\Omega_{\{\rho_m,\lambda_m\}}|\,\Phi(t,\theta)\,\Phi(t',\theta')\,|\Omega_{\{\rho_m,\lambda_m\}}\rangle \\ &= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\tilde{u}_m(t)\,\tilde{u}_m^*(t')\,e^{im(\theta-\theta')} \end{aligned} \tag{7.12}$$

is not a function of dS invariant quantities. The requirement that the vacuum be dS invariant puts severe constraints on the $\tilde{u}_m$. Since all the dS invariant vacua are related to each others by Bogoliubov transformation (7.5), this requirement constrains ρ_m to be m independent and λ_m to possess a given m dependence associated with the phase of the modes u_m.

If we choose the reference vacuum to be the Bunch-Davies vacuum, the time dependence at fixed m is

$$\begin{aligned} u_m(t)u_m^*(t') &= \cosh^2\rho_m\, u_m^{\rm BD}(t)u_{-m}^{\rm BD*}(t') + \sinh^2\rho_m u_m^{\rm BD*}(t)u_{-m}^{\rm BD}(t') \\ &\quad + \sinh\rho_m\cosh\rho_m\Big(e^{i\lambda_m}\, u_m^{\rm BD}(t)u_{-m}^{\rm BD}(t') + e^{-i\lambda_m}u_m^{\rm BD*}(t)u_{-m}^{\rm BD*}(t')\Big)\,. \end{aligned} \tag{7.13}$$

Then, if as in [11], we choose the arbitrary phase by imposing

$$u_m^{\rm BD*}(t) = (-1)^m\, u_m^{\rm BD}(-t)\,, \tag{7.14}$$

taking $\rho_m = \alpha$ and $\lambda_m = \beta$, the two point function (7.12) evaluated in this (α,β)-vacuum coincides with the expression in eq.(5.17).

In the formalism adopted in this paper, dS invariance is built in, and the field operators are decomposed in Fourier modes given in eq.(6.5). Using the time reversal invariance of the BD field operator as well as the explicit expression of the time reversal operators given in eq.(4.48), we obtain

$$u_m^{\rm BD}(-t) = \frac{\gamma_m^*}{\gamma_m}\, u_m^{\rm BD*}(t)\,. \tag{7.15}$$

When taking $\rho_m = \alpha$ and $e^{i\lambda_m} = e^{i\beta}(-1)^m\frac{\gamma_m}{\gamma_m^*}$, the two-point function (7.12) gives again (5.17). However, because of dS invariance, the normalization coefficient of eq.(7.11) vanishes. Different vacua are therefore only formally related by the operator $\mathcal{S}_{\alpha,\beta} = \prod_{m=0}^{\infty}\mathcal{S}_{\{\alpha,\beta\}}$. In fact, the different vacua belong to inequivalent Fock spaces.

Proceeding formally, the action of this operator $\mathcal{S}_{\alpha,\beta}$ on the field $\Phi_{\rm BD}$ gives

$$\mathcal{S}^{\dagger}_{\alpha,\beta}\,\Phi_{\rm BD}(0)\,\mathcal{S}_{\alpha,\beta} = \sum_{m=-\infty}^{\infty} c_m^{\rm BD}\,\mathcal{S}^{\dagger}_{\alpha,\beta}\,a^{\dagger}_m\,\mathcal{S}_{\alpha,\beta} + c_m^{\rm BD*}\,\mathcal{S}^{\dagger}_{\alpha,\beta}\,a_m\,\mathcal{S}_{\alpha,\beta}$$

$$= \sum_{m=-\infty}^{\infty} c_m^{\rm BD}\left(\cosh\alpha\, a^{\dagger}_m + e^{-i\beta}(-1)^m\frac{\gamma^*_m}{\gamma_m}\sinh\alpha\, a_m\right) + \text{h.c.} \tag{7.16}$$

From eq.(5.13), we have the relation between general coefficients c_m and $c_m^{\rm BD}$:

$$c_m = c_m^{\rm BD}(\cosh\alpha + (-1)^m e^{i\beta}\sinh\alpha)\,. \tag{7.17}$$

This implies that

$$\mathcal{S}^{\dagger}_{\alpha,\beta}\,\Phi_{\rm BD}(0)\,\mathcal{S}_{\alpha,\beta} = \Phi_{\alpha,\beta}(0)\,. \tag{7.18}$$

Moreover since the squeezing operator commutes with all generators, we have, at every space-time point,

$$\mathcal{S}^{\dagger}_{\alpha,\beta}\,\Phi_{\rm BD}(x)\,\mathcal{S}_{\alpha,\beta} = \Phi_{\alpha,\beta}(x)\,. \tag{7.19}$$

This establishes in full generality the relation between the usual QFT treatment and our approach. Indeed, collecting the various results, we have

$$\begin{aligned} G_{\alpha,\beta}(x,x') &= \langle\Omega|\,\Phi_{\alpha,\beta}(x)\,\Phi_{\alpha,\beta}(x')\,|\Omega\rangle, \\ &= \langle\Omega|\,\mathcal{S}^{\dagger}_{\alpha,\beta}\,\Phi_{\rm BD}(x)\,\Phi_{\rm BD}(x')\,\mathcal{S}_{\alpha,\beta}|\Omega\rangle, \\ &= \langle\Omega_{\alpha,\beta}|\,\Phi_{\rm BD}(x)\,\Phi_{\rm BD}(x')\,|\Omega_{\alpha,\beta}\rangle\,, \end{aligned} \tag{7.20}$$

where $|\Omega_{\alpha,\beta}\rangle$ is the α-vacuum conventionally defined.

8. Arbitrary dimensions

In this section we shall show that the approach we have used in two dimensions can be generalized to arbitrary dimensions. In particular, when considering the principal series, that is, for scalar fields with mass squared greater than $(n-1)^2/4$, we shall see that the moduli space of covariant canonical fields stays $\mathrm{SU}(1,1)/\mathrm{U}(1)$ irrespectively of the dimensionality. On the contrary the *in-out* Bogoliubov coefficients do depend on the dimensionality: they vanish for all odd dimensional dS spaces.

The n-dimensional de Sitter space, dS_n is described by the hyperboloid in $(n+1)$-dimensional Minkowski space $\mathbb{R}^{1,n}$:

$$\eta_{AB}X^A X^B = 1\,, \tag{8.1}$$

where $\eta_{AB} = diag(-1,1,\ldots,1)$. In the following we use the index notation:

$$\begin{aligned} &A,B,C,D = 0,1,\ldots,n\,; \quad && I,J = 1,2,\ldots,n\,; \\ &\mu,\nu = 0,1,\ldots,n-1\,; && i,j,k = 1,2,\ldots,n-1\,. \end{aligned} \tag{8.2}$$

8.1 The $\mathrm{SO}_0(1,n)$ group

The isometry group of dS_n is $\mathrm{SO}_0(1,n)$. It is the group of transformations continuously connected to the identity which leaves eq.(8.1) invariant. The generators of $\mathrm{SO}_0(1,n)$ verify the following algebra:

$$[\,\mathcal{M}_{AB},\,\mathcal{M}_{CD}\,] \;=\; -i\;(\,\eta_{AC}\mathcal{M}_{BD} - \eta_{AD}\mathcal{M}_{BC} - \eta_{BC}\mathcal{M}_{AD} + \eta_{BD}\mathcal{M}_{AC}\,)\,. \tag{8.3}$$

When considering the principal series, the UIR of $\mathrm{SO}_0(1,n)$ can be realized by the square integrable functions on S^{n-1} : $\mathcal{L}^2(S^{n-1})$. The $(n-1)$-sphere is conventionally parameterized by a vector $\vec{\zeta}$ in $\mathbb{R}^n$ subject to $|\vec{\zeta}| = 1$. The action of the generators of $\mathrm{SO}_0(1,n)$ on the UIR states are given by

$$\begin{aligned}
\mathcal{M}_{IJ} &= i\left(\zeta_I\frac{\partial}{\partial\zeta^J} - \zeta_J\frac{\partial}{\partial\zeta^I}\right)\,,\\
\mathcal{M}_{I0} &= \frac{1}{2}\left(\mathcal{M}_{IJ}\zeta^J + \zeta^J\mathcal{M}_{IJ}\right) + \mu\,\zeta_I\\
&= \zeta^J\mathcal{M}_{IJ} + \left(\mu + i\frac{n-1}{2}\right)\zeta_I\,,
\end{aligned} \tag{8.4}$$

and the quadratic Casimir is given by

$$\mathcal{C} \;=\; \frac{1}{2}\sum_{A,B}\mathcal{M}_{AB}\mathcal{M}^{AB} \;=\; -\mu^2 - \frac{(n-1)^2}{2}\,. \tag{8.5}$$

It would be useful to have the action under finite transformations, we have

$$\begin{aligned}
\langle\,\vec{\zeta}\,|\,e^{i\theta\mathcal{M}^{IJ}}\,|\Psi\rangle \;&=\; \langle\,\vec{\zeta}'\,|\Psi\rangle\,,\\
\langle\,\vec{\zeta}\,|\,e^{i\omega\mathcal{M}^{I0}}\,|\Psi\rangle \;&=\; (\cosh\omega + \zeta^I\sinh\omega)^{i\mu-\frac{n-1}{2}}\langle\,\vec{\zeta}''\,|\Psi\rangle\,,
\end{aligned} \tag{8.6}$$

where

$$\begin{aligned}
\vec{\zeta}' &= e^{i\theta\mathscr{M}^{IJ}}\vec{\zeta}\,,\\
\vec{\zeta}'' &= \left(\frac{\zeta^1}{\cosh\omega + \zeta^I\sinh\omega},\,\ldots\,,\frac{\sinh\omega + \zeta^I\cosh\omega}{\cosh\omega + \zeta^I\sinh\omega},\,\ldots\,,\frac{\zeta^n}{\cosh\omega + \zeta^I\sinh\omega}\right),
\end{aligned} \tag{8.7}$$

and where $\mathscr{M}^{IJ}$ is the representation of $\mathcal{M}^{IJ}$ on the vector space $\mathbb{R}^n$. The inner product of two position eigenstates is

$$\langle\,\vec{\zeta}\,|\,\vec{\zeta}'\,\rangle = \delta^{n-1}(\vec{\zeta} - \vec{\zeta}')\,, \tag{8.8}$$

where the delta function $\delta^{n-1}(\vec{\zeta})$ is defined on S^{n-1}.

8.2 Massive scalar field in dS_n

We choose the origin in dS_n to have the embedding coordinates $X_0^A = (0,0,\ldots,0,1)$. The field on any point of dS_n, of coordinates $X^A = \Lambda^A{}_B X_0^B$, can be deduced from the field at the origin $\Phi(X_0)$ by

$$\Phi(X) = U(\Lambda)\,\Phi(X_0)\,U(\Lambda)^{-1}\,, \tag{8.9}$$

where Λ is an element of $\mathrm{SO}_0(1,n)$.

In the global coordinate system:

$$\begin{aligned} X^0 &= \sinh t\,, \\ X^I &= \cosh t\,\xi^I\,, \qquad \vec{\xi}\in S^{n-1}\,, \\ \xi^I &= \mathrm{R}^I{}_J\,\xi_0^J\,, \qquad \vec{\xi}_0 = (0,\ldots,0,1)\,, \end{aligned} \tag{8.10}$$

where R is a element of $\mathrm{SO}(n)$ subgroup. The metric reads

$$ds^2 = -dt^2 + \cosh^2 t\, d\Omega^2(\vec{\xi})\,. \tag{8.11}$$

Therefore the point X_0 can be transported to any point X by a boost followed by a rotation. This implies that eq.(8.9) can be written as

$$\Phi(X) = U(\mathrm{R})\, e^{itM^{0n}}\,\Phi(X_0)\,e^{-itM^{0n}}\,U(\mathrm{R})^{-1}\,. \tag{8.12}$$

The origin $X_0 = (0,\vec{\xi}_0)$ is invariant under the action of the subgroup $\mathrm{SO}_0(1,n-1)$ generated by $M_{\mu\nu}$. To construct a local field, we thus require

$$[\,M_{\mu\nu},\,\Phi(0,\vec{\xi}_0)\,] = 0\,. \tag{8.13}$$

As in two dimensions, after some algebra, one sees that the Casimir relation gives the Klein-Gordon equation:

$$\left(\partial_t^2 + (n-1)\tanh t\,\partial_t - \frac{1}{\cosh^2 t}\,\Delta_{\vec{\xi}}\right)\Phi(t,\vec{\xi}) = -\left(\mu^2 + \frac{(n-1)^2}{4}\right)\Phi(t,\vec{\xi})\,, \tag{8.14}$$

where $\Delta_{\vec{\xi}}$ is the Laplacian on S^{n-1}.

From the UIR of $\mathrm{SO}_0(1,n)$ we define the creation and annihilation operators by

$$\begin{gathered} a^\dagger(\vec{\zeta})|\Omega\rangle = |\,\vec{\zeta}\,\rangle\,, \\ [\,a(\vec{\zeta}),\,a^\dagger(\vec{\zeta}')\,] = \delta^{n-1}(\vec{\zeta}-\vec{\zeta}')\,, \qquad [\,a(\vec{\zeta}),\,a(\vec{\zeta}')\,] = 0\,. \end{gathered} \tag{8.15}$$

The field operator in the origin can be expanded in terms of these operators:

$$\Phi(0,\vec{\xi}_0) = \int d^{n-1}\Omega(\vec{\zeta})\ \left[\Psi_0(\vec{\zeta})\,a^\dagger(\vec{\zeta}) \;+\; \Psi_0^*(\vec{\zeta})\,a(\vec{\zeta})\right], \tag{8.16}$$

where $d^{n-1}\Omega(\vec{\zeta})$ is the invariant volume element on S^{n-1}.

The covariance condition (8.13) determines the function $\Psi_0(\vec{\zeta})$. The rotation part of this equation implies that $\Psi_0(\vec{\zeta})$ depends on $\vec{\zeta}$ only through $\vec{\zeta}\cdot\vec{\xi}_0$, whereas the boost part fixes this dependence to be again governed by two arbitrary coefficients:

$$\begin{aligned} \Psi_0(\vec{\zeta}) &= A\,\Theta(\vec{\zeta}\cdot\vec{\xi}_0)\,\left(\vec{\zeta}\cdot\vec{\xi}_0\right)^{i\mu-\frac{n-1}{2}} + B\,\Theta(-\vec{\zeta}\cdot\vec{\xi}_0)\,\left(-\vec{\zeta}\cdot\vec{\xi}_0\right)^{i\mu-\frac{n-1}{2}} \\ &= C\,(\vec{\zeta}\cdot\vec{\xi}_0 - i\epsilon)^{i\mu-\frac{n-1}{2}} + D\,(\vec{\zeta}\cdot\vec{\xi}_0 + i\epsilon)^{i\mu-\frac{n-1}{2}}\,, \end{aligned} \tag{8.17}$$

where the limit $\epsilon \to 0^+$ is understood. We have also the following relation between the coefficients:

$$A = C + D\,, \qquad B = (e^{-i\pi})^{i\mu - \frac{n-1}{2}}\, C + (e^{i\pi})^{i\mu - \frac{n-1}{2}}\, D\,, \tag{8.18}$$

which generalize eq.(4.21). Transporting the field with a boost followed by a rotation brings the point $(0, \vec{\xi}_0)$ to $(t, \vec{\xi})$. We thus get

$$\Phi(t, \vec{\xi}) \;=\; \int d^{n-1}\Omega(\vec{\zeta})\;\, \Psi_{t,\vec{\xi}}(\vec{\zeta})\, a^\dagger(\vec{\zeta}) \;+\; \Psi^*_{t,\vec{\xi}}(\vec{\zeta})\, a(\vec{\zeta})\,, \tag{8.19}$$

with

$$\Psi_{t,\vec{\xi}}(\vec{\zeta}) \;=\; C\,(\cosh t\, \vec{\zeta}\cdot\vec{\xi} + \sinh t - i\epsilon)^{i\mu - \frac{n-1}{2}} + D\,(\cosh t\, \vec{\zeta}\cdot\vec{\xi} + \sinh t + i\epsilon)^{i\mu - \frac{n-1}{2}}\,. \tag{8.20}$$

The two point function is simply expressed in terms of $\Psi_{t,\xi}(\vec{\zeta})$ as

$$\langle\Omega|\, \Phi(t, \vec{\xi})\, \Phi(t', \vec{\xi}')\, |\Omega\rangle \;=\; \int d^{n-1}\Omega(\vec{\zeta})\;\, \Psi^*_{t,\vec{\xi}}(\vec{\zeta})\, \Psi_{t',\vec{\xi}'}(\vec{\zeta})\,. \tag{8.21}$$

The above integral can be expressed in terms of hypergeometric functions as shown in the Appendix A. We obtain

$$\begin{aligned}\langle\Omega|\, \Phi(x)\, \Phi(x')\, |\Omega\rangle \;=\; & |C|^2 e^{\pi\mu}\, F_n(x; x') \;+\; |D|^2 e^{-\pi\mu}\, F_n(\bar{x}; \bar{x}') \\ & + 2\,\mathrm{Re}\left[C^* D\, e^{-i\frac{n-1}{2}\pi}\, F_n(x; \bar{x}')\right],\end{aligned} \tag{8.22}$$

with

$$F_n(x; x') = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\, {}_2F_1\left(i\mu + \frac{n-1}{2}, -i\mu + \frac{n-1}{2}; \frac{n}{2}; \frac{1 + \tilde{Z}(x; x')}{2}\right). \tag{8.23}$$

We have defined $\tilde{Z}$ by

$$\tilde{Z}(x; x') = Z(x; x') + i\,\mathrm{sgn}(t - t')\,\epsilon\,, \tag{8.24}$$

where Z is the dS invariant quantity:

$$Z(t, \vec{\xi}; t', \vec{\xi}') = \cosh t \cosh t'\, \vec{\xi}\cdot\vec{\xi}' - \sinh t \sinh t' = X^A X'_A\,. \tag{8.25}$$

Because of the $i\epsilon$ term, $\tilde{Z}$ is not a dS invariant for $|Z| < 1$. Nevertheless the two point function is dS invariant. The reason is the following. The hypergeometric function has a branch cut for $Z > 1$. Therefore the $i\epsilon$ term is irrelevant for $Z < 1$. When $Z > 1$ instead, it becomes relevant but $\mathrm{sgn}(t - t') = \mathrm{sgn}(X^0 - X'^0)$ and hence $\tilde{Z}$ are dS invariant. It should be also emphasized that the $i\epsilon$ term in eq.(8.22) originates from eq.(8.17). Therefore what we obtain is the positive Wightman function.

As in the two dimensional case, the reality of the two point function for space-like separation, or the parity argument, insures that the equal time commutator of two fields vanishes.

Proceeding as in Section 4, it can be easily seen that the canonical commutation relation is satisfied modulo a constant which should be identified with $\hbar$. More explicitly we have

$$\left[\partial_t \Phi(0, \vec{\xi}_0),\; \Phi(0, \vec{\xi})\right] = i N_n\, \delta^{n-1}(\vec{\xi}_0 - \vec{\xi})\,, \tag{8.26}$$

with

$$N_n = 2\,\mathrm{Im}\left[\int d^{n-1}\Omega(\vec{\xi}) \int d^{n-1}\Omega(\vec{\zeta})\;\partial_t \Psi^*_{0,\vec{\xi}_0}(\vec{\zeta})\,\Psi_{0,\vec{\xi}}(\vec{\zeta})\right]. \tag{8.27}$$

Using the explicit form of $\Psi_{t,\vec{\xi}}(\vec{\zeta})$, the integral can be calculated, and the canonical commutation relation leads to

$$-e^{\pi\mu}|C|^2 + e^{-\pi\mu}|D|^2 = 2^{-n-1}\pi^{-n}\left|\Gamma\left(i\mu + \frac{n-1}{2}\right)\right|^2 . \tag{8.28}$$

By defining the vector:

$$\mathbf{z} = \frac{2^{\frac{n+1}{2}}\pi^{\frac{n}{2}}}{\left|\Gamma\left(i\mu+\frac{n-1}{2}\right)\right|}\begin{pmatrix} e^{-\frac{\pi\mu}{2}}D \\ e^{\frac{\pi\mu}{2}}e^{i\frac{n-1}{2}\pi}C\end{pmatrix}, \tag{8.29}$$

the canonical condition implies eq.(4.44), therefore its solutions are still of the form eq.(4.45).

If we now impose the time reversal symmetry on the field, $\mathsf{T}\Phi(t,\vec{\xi})\mathsf{T}^{-1} = \Phi(-t,\vec{\xi})$, we get a constraint for the coefficients C and D.

As in the two dimensional case, the *unitary* time reversal operator is not compatible with the canonical commutation relation (8.28), since it gives

$$e^{\pi\mu}\,|C|^2 \;=\; e^{-\pi\mu}\,|D|^2 . \tag{8.30}$$

With the *anti-unitary* time reversal operator instead, we get

$$\begin{aligned}\langle\Omega|\,\Phi(t,\vec{\xi})\,\Phi(t',\vec{\xi}')\,|\Omega\rangle \;&=\; \langle\Omega|\mathsf{T}^{-1}\,\mathsf{T}\Phi(t,\vec{\xi})\mathsf{T}^{-1}\,\mathsf{T}\Phi(t',\vec{\xi}')\mathsf{T}^{-1}\,\mathsf{T}|\Omega\rangle \\ &=\; \langle\Omega|\,\Phi(-t,\vec{\xi})\,\Phi(-t',\vec{\xi}')\,|\Omega\rangle^* .\end{aligned} \tag{8.31}$$

This gives a constraint which is compatible with the canonical commutation relation:

$$e^{-i\frac{n-1}{2}\pi}\,C^*\,D \;=\; e^{i\frac{n-1}{2}\pi}\,C\,D^* . \tag{8.32}$$

In terms of the vector $\mathbf{z}$, the above relation reads

$$\mathbf{z}^\dagger\sigma_2\,\mathbf{z} = 0\,, \tag{8.33}$$

which implies $\beta = 0$, as before.

8.3 Bunch-Davies vacuum and *in* and *out* vacua

In the coincidence point limit, the Minkowski vacuum positive Wightman function behaves as

$$G_{\mathrm{Mink}}(x;x') \underset{x\to x'}{\approx} \frac{1}{4\pi^{\frac{n}{2}}} \times \begin{cases} \log|\,(x-x')^2 - (t-t'-i\epsilon)^2\,|^{-2}\,, & \text{for } n=2\,, \\ \\ \Gamma(\frac{n}{2}-1)|\,(\vec{x}-\vec{x}')^2 - (t-t'-i\epsilon)^2\,|^{\frac{2-n}{2}}\,, & \text{for } n\neq 2\,. \end{cases} \tag{8.34}$$

The short distance behavior of two point function in dS_n, eq.(8.22), is

$$G_{dS_n}(x;x') \underset{x\to x'}{\approx} \frac{\pi^{\frac{n}{2}}2^{n-1}}{\left|\Gamma\left(i\mu+\frac{n-1}{2}\right)\right|^2}\times\begin{cases} |C|^2e^{\pi\mu}\,\log|\,(x-x')^2-(t-t'+i\epsilon)^2\,|^{-2} \\ +\,|D|^2e^{-\pi\mu}\,\log|\,(x-x')^2-(t-t'-i\epsilon)^2\,|^{-2}\,, & \text{for } n=2\,, \\ \\ |C|^2e^{\pi\mu}\,|\,(\vec{x}-\vec{x}')^2-(t-t'+i\epsilon)^2\,|^{\frac{2-n}{2}} \\ +\,|D|^2e^{-\pi\mu}\,|\,(\vec{x}-\vec{x}')^2-(t-t'-i\epsilon)^2\,|^{\frac{2-n}{2}}\,, & \text{for } n\neq 2\,. \end{cases} \tag{8.35}$$

This has been obtained from the asymptotic behavior of the hypergeometric function presented in Appendix A. Imposing that the behavior be that of Hadamard, one gets the Bunch-Davies coefficients:

$$C^{\mathrm{BD}}=0\,,\qquad D^{\mathrm{BD}}=\frac{e^{\frac{\pi\mu}{2}}}{2^{\frac{n+1}{2}}\pi^{\frac{n}{2}}}\left|\Gamma\left(i\mu+\frac{n-1}{2}\right)\right|. \tag{8.36}$$

In terms of $\mathbf{z}$, the Bunch-Davies vacuum still corresponds to $\mathbf{z}^{\mathrm{BD}}=\binom{1}{0}$, and the general (C,D) vacuum can be obtained from the Bunch-Davies coefficients:

$$C=e^{i\beta}\sinh\alpha\, e^{-\pi(\mu+i\frac{n-1}{2})}\,D^{\mathrm{BD}}\,,\qquad D=\cosh\alpha\, D^{\mathrm{BD}}\,. \tag{8.37}$$

The behavior of the field in asymptotic past and future can be extracted from that of the two point function. We put a field operator in the far past (or future) with coefficients C and D and another field at the origin with different coefficients $\tilde{C}$ and $\tilde{D}$ to avoid any confusion. We thus analyze

$$\langle\Omega|\,\Phi_{\tilde{C},\tilde{D}}(0,\vec{\xi_0})\,\Phi_{C,D}(t,\vec{\xi_0})\,|\Omega\rangle\,, \tag{8.38}$$

in the limit $t\to\pm\infty$. From the asymptotic expression of the hypergeometric function given in Appendix A, we can see that the two point function has the following behavior:

$$\begin{aligned} &\langle\Omega|\,\Phi_{\tilde{C},\tilde{D}}(0,\vec{\xi_0})\,\Phi_{C,D}(t,\vec{\xi_0})\,|\Omega\rangle \underset{t\to\pm\infty}{\approx} \frac{2^n\pi^{\frac{n}{2}}}{\left|\Gamma\left(i\mu+\frac{n-1}{2}\right)\right|^2}\times \\ &\times\Bigg[\,2^{2i\mu}\Gamma(-2i\mu)\Big\{\tilde{C}^*C\,e^{\pi\mu}\,(-e^{\pm t}\pm i\epsilon)^{-i\mu-\frac{n-1}{2}}+\,\tilde{D}^*D\,e^{-\pi\mu}\,(-e^{\pm t}\mp i\epsilon)^{-i\mu-\frac{n-1}{2}} \\ &\qquad\qquad +\tilde{C}^*D\,e^{-i\frac{n-1}{2}\pi}\,(e^{\pm t}\mp i\epsilon)^{-i\mu-\frac{n-1}{2}}+\,\tilde{D}^*C\,e^{i\frac{n-1}{2}\pi}\,(e^{\pm t}\pm i\epsilon)^{-i\mu-\frac{n-1}{2}}\Big\} \\ &\quad +2^{-2i\mu}\Gamma(2i\mu)\Big\{\tilde{C}^*C\,e^{\pi\mu}\,(-e^{\pm t}\pm i\epsilon)^{i\mu-\frac{n-1}{2}}+\,\tilde{D}^*D\,e^{-\pi\mu}\,(-e^{\pm t}\mp i\epsilon)^{i\mu-\frac{n-1}{2}} \\ &\qquad\qquad +\tilde{C}^*D\,e^{-i\frac{n-1}{2}\pi}\,(e^{\pm t}\mp i\epsilon)^{i\mu-\frac{n-1}{2}}+\,\tilde{D}^*C\,e^{i\frac{n-1}{2}\pi}\,(e^{\pm t}\pm i\epsilon)^{i\mu-\frac{n-1}{2}}\Big\}\Bigg]\,. \end{aligned} \tag{8.39}$$

In remote past we have a negative frequency term proportional to

$$\left(e^{\pi\mu}\,\tilde{C}^*+e^{-\pi\mu}\,\tilde{D}^*\right)\left(e^{\pi(\mu+i\frac{n-1}{2})}\,C+\,e^{-\pi(\mu+i\frac{n-1}{2})}\,D\right)\,e^{-i\mu t}\,. \tag{8.40}$$

The *in* vacuum thus corresponds to

$$e^{\pi(\mu+i\frac{n-1}{2})}\,C^{\mathrm{IN}}+e^{-\pi(\mu+i\frac{n-1}{2})}\,D^{\mathrm{IN}}=0\,. \tag{8.41}$$

Similarly, in the far future we have

$$\Big(e^{\pi(\mu-i\frac{n-1}{2})}\,\tilde{C}^{*}+e^{-\pi(\mu-i\frac{n-1}{2})}\,\tilde{D}^{*}\Big)\Big(e^{\pi\mu}\,C+e^{-\pi\mu}\,D\Big)\,e^{-i\mu t}\,, \tag{8.42}$$

and the *out* vacuum corresponds to

$$e^{\pi\mu}\,C^{\mathrm{OUT}}+e^{-\pi\mu}\,D^{\mathrm{OUT}}=0\,. \tag{8.43}$$

Notice that for odd dimensions, we have the same condition for the *in* and the *out* vacuum. This implies that there is no particle creation in odd dimensions for all scalar fields belonging to the principal series. It will be interesting to further investigate this phenomenon.

In terms of A and B the above conditions read

$$\begin{aligned} B^{\mathrm{IN}} &= 0\,,\\ \sin(\tfrac{n-1}{2}\pi)\,A^{\mathrm{OUT}} &= i\,\sinh\pi\mu\,B^{\mathrm{OUT}}\,, \end{aligned} \tag{8.44}$$

which clearly show that $B^{\mathrm{IN}}=B^{\mathrm{OUT}}=0$ when n is odd. As in two dimensions therefore we find that the *in* vacuum is characterized by a state $|\Psi_0\rangle$ in the UIR whose support is inside the past horizon of $(t=-\infty,\vec{\xi}_0)$, see eq.(8.17).

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A. Wightman function in arbitrary dimension

In this Appendix we compute the Wightman function in n-dimensions. In the treatment we adopted, the $i\epsilon$ prescription follows from the writing of the vector state $|\Psi_0\rangle$ in terms of the holomorphic and anti-holomorphic sectors weighted by the coefficients C and D respectively, see eq.(8.20). Instead, in the usual approach of quantum fields in dS, see [11], the $i\epsilon$ prescription is ambiguous: one obtains the Wightman function by first solving the Klein-Gordon equation assuming that it depends only on the invariant quantity $Z=\eta_{AB}X^AY^B$. The ambiguity arises because Z is invariant under the exchange of the two points and because Wightman functions are sensitive to the ordering for time-like separated points. In order to cure this problem, one introduces from the outset an imaginary prescription (an $i\epsilon$ term) to Z, the sign of which is that of $X^0-X'^0$, see eq.(8.24).

As explained in section 8, the two point function in n-dimensions can be obtained as

$$G(t,\vec{\xi};t',\vec{\xi}')\;=\;\langle\Omega|\,\Phi(t,\vec{\xi})\,\Phi(t',\vec{\xi}')\,|\Omega\rangle\;=\;\int d^{n-1}\Omega(\vec{\zeta})\,\Psi^{*}_{t,\vec{\xi}}(\vec{\zeta})\Psi_{t',\vec{\xi}'}(\vec{\zeta})\,, \tag{A.1}$$

with $\Psi_{t,\vec{\xi}}(\vec{\zeta})$ given in eq.(8.20). Each term with coefficients $|C|^2$, $|D|^2$, C^*D and D^*C can be written in terms of a single function f with appropriate arguments as

$$\begin{aligned} G(x;x') &= |C|^2\, f(x;x') \,+\, |D|^2 e^{-2\pi\mu}\, f(\bar{x};\bar{x}') \\ &\quad + C^* D\, e^{-\pi\mu} e^{-i\frac{n-1}{2}\pi}\, f(x;\bar{x}') \,+\, D^* C e^{-\pi\mu} e^{i\frac{n-1}{2}\pi}\, f(\bar{x};x')\,, \end{aligned} \tag{A.2}$$

with

$$f(t,\vec{\xi};t',\vec{\xi}') = \int d^{n-1}\Omega(\vec{\zeta})\,(\cosh t\,\vec{\zeta}\cdot\vec{\xi} + \sinh t + i\epsilon)^{-i\mu-\frac{n-1}{2}}\,(\cosh t'\,\vec{\zeta}\cdot\vec{\xi}' + \sinh t' - i\epsilon)^{i\mu-\frac{n-1}{2}}\,. \tag{A.3}$$

Using the following identity, valid for positive ϵ:

$$\frac{1}{(A+i\epsilon)^\alpha (B-i\epsilon)^\beta} = e^{i\pi\beta}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 dx\int_0^1 dy\,\delta(x+y-1)\frac{x^{\alpha-1}y^{\beta-1}}{(xA - yB + i\epsilon)^{\alpha+\beta}}\,, \tag{A.4}$$

the integral in eq.(A.3) can be put in the form:

$$\begin{aligned} &\frac{e^{\pi\mu}\, e^{i\frac{n-1}{2}\pi}\,\Gamma(n-1)}{\Gamma\left(i\mu+\frac{n-1}{2}\right)\Gamma\left(-i\mu+\frac{n-1}{2}\right)}\int_0^1 dx\int_0^1 dy\,\delta(x+y-1)\,x^{-i\mu+\frac{n-3}{2}}y^{i\mu+\frac{n-3}{2}} \\ &\times\int d^{n-1}\Omega(\vec{\zeta})\left(\vec{\zeta}\cdot(x\cosh t\,\vec{\xi} - y\cosh t'\,\vec{\xi}') + (x\sinh t - y\sinh t') + i\epsilon\right)^{1-n}. \end{aligned} \tag{A.5}$$

Introducing $\vec{a}$ and b by

$$\vec{a} = x\cosh t\,\vec{\xi} - y\cosh t'\,\vec{\xi}'\,, \qquad b = x\sinh t - y\sinh t'\,, \tag{A.6}$$

and defining ϕ as the angle between the vector $\vec{\zeta}$ and $\vec{a}$, the integral with respect to $\vec{\zeta}$ on S^{n-1} reads

$$\int_0^\pi d\phi\,\sin^{n-2}\phi\left(|\vec{a}|\cos\phi + b + i\epsilon\right)^{1-n}\int d^{n-2}\Omega\,, \tag{A.7}$$

where the integral over S^{n-2} gives the area $2\pi^{\frac{n-1}{2}}/\Gamma(\frac{n-1}{2})$. The remaining integral can be calculated as

$$\begin{aligned} &\textstyle\int_0^\pi d\phi\,\sin^{n-2}\phi\,(|\vec{a}|\cos\phi + b + i\epsilon)^{1-n} \,=\, (-1)^{n-1}\int_0^\pi d\phi\,\sin^{n-2}\phi\left(|\vec{a}| + (b+i\epsilon)\cos\phi\right)^{1-n} \\ &= (-1)^{n-1}\pi^{1/2}\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}(|\vec{a}|^2 - b^2 - 2\,i\epsilon\, b)^{\frac{1-n}{2}} \,=\, (-1)^{n-1}\pi^{1/2}\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}(x^2+y^2-2xy\,\tilde{Z})^{\frac{1-n}{2}}\,, \end{aligned} \tag{A.8}$$

where $\tilde{Z}$ is the time ordered invariant distance:

$$\tilde{Z}(t,\vec{\xi};t',\vec{\xi}') = \cosh t\cosh t'\,\vec{\xi}\cdot\vec{\xi}' - \sinh t\sinh t' + i\,\mathrm{sgn}(t-t')\,\epsilon\,. \tag{A.9}$$

Finally, the integral with respect to x and y gives a hypergeometric function:

$$\begin{aligned} f \,&=\, e^{\pi\mu}\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\frac{\Gamma(n-1)}{\Gamma\left(i\mu+\frac{n-1}{2}\right)\Gamma\left(-i\mu+\frac{n-1}{2}\right)} \\ &\qquad\times\int_0^1 dx\int_0^1 dy\,\delta(x+y-1)\,x^{-i\mu+\frac{n-3}{2}}y^{i\mu+\frac{n-3}{2}}(x^2+y^2-2xy\tilde{Z})^{1-n}, \\ &= e^{\pi\mu}\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\,{}_2F_1\left(i\mu+\frac{n-1}{2}, -i\mu+\frac{n-1}{2}; \frac{n}{2}; \frac{1+\tilde{Z}}{2}\right). \end{aligned} \tag{A.10}$$

If we now substitute in eq.(A.2), the obtained expression of f we get eq.(8.22) used in the text.

To conclude this Appendix, we present two formula used in section 8. The small distance behavior of the two-point is governed by the limit given in [39]:

$$ {}_2F_1\left(i\mu+\frac{n-1}{2},-i\mu+\frac{n-1}{2};\frac{n}{2};\frac{1+x}{2}\right)\underset{x\to 1}{\approx}\begin{cases}\frac{-2}{\left|\Gamma\left(i\mu+\frac{1}{2}\right)\right|^2}\log(1-x)\,, & \text{for } n=2\,,\\[2ex] \frac{2^{n-2}\,\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{2}-1\right)}{\left|\Gamma\left(i\mu+\frac{n-1}{2}\right)\right|^2}(1-x)^{2-n}\,, & \text{for } n\neq 2\,.\end{cases} \tag{A.11} $$

In subsection 8.3, we used the asymptotic behavior of the two-point. It is governed by [39]:

$$ \begin{aligned} &{}_2F_1\left(i\mu+\frac{n-1}{2},-i\mu+\frac{n-1}{2};\frac{n}{2};\frac{1+x}{2}\right)\\ &\underset{x\to\infty}{\approx}\frac{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)}{\left|\Gamma\left(i\mu+\frac{n-1}{2}\right)\right|^2}\left[2^{i\mu}\Gamma(-2i\mu)\,(-x)^{-i\mu-\frac{n-1}{2}}+2^{-i\mu}\Gamma(2i\mu)\,(-x)^{i\mu-\frac{n-1}{2}}\right]. \end{aligned} \tag{A.12} $$

B. Flat sections

In this Appendix we consider the quantization of a massive scalar field using flat sections. This amounts to use the basis of the UIR which is adapted to the isometry group of these. Since the state $|\Psi_0\rangle$ is unchanged, its components in the new basis give the expansion of the field operator in terms of the new creation and annihilation operators.

We first recall the flat section parameterization:

$$ \begin{aligned} X^0 &= -\sinh\tau-\frac{1}{2}e^{\tau}\sum_{i=1}^{n-1}(x^i)^2\,,\\ X^i &= e^{\tau}x^i\,,\\ X^n &= \cosh\tau-\frac{1}{2}e^{\tau}\sum_{i=1}^{n-1}(x^i)^2\,. \end{aligned} \tag{B.1} $$

It covers the $X^n > X^0$ part of dS space and leads to the metric:

$$ ds^2=-d\tau^2+e^{2\tau}\sum_{i=1}^{n-1}(dx^i)^2\,, \tag{B.2} $$

showing that the isometry group of the constant τ sections is the $(n-1)$-dimensional Euclidean group. The infinitesimal action of the Killing vector fields is given by

$$ \begin{array}{ll} \mathcal{M}_{n0} & :\ \delta\tau=\epsilon\,,\quad \delta\vec{x}=-\epsilon\vec{x}\,,\\ \mathcal{P}_i=\frac{1}{\sqrt{2}}(\mathcal{M}_{i0}-\mathcal{M}_{in}) & :\ \delta\tau=0\,,\quad \delta x^i=\epsilon\,,\\ \mathcal{M}_{ij} & :\ \delta\tau=0\,,\quad \delta x^i=\epsilon\,,\ \delta x^j=-\epsilon\,,\\ \frac{1}{\sqrt{2}}(\mathcal{M}_{i0}+\mathcal{M}_{in}) & :\ \delta\tau=\epsilon x^i\,,\ \delta x^i=-\frac{\epsilon}{2}\left((x^i)^2+e^{-2\tau}\right). \end{array} \tag{B.3} $$

These allow to identify $\mathcal{P}_i$ as the translation generators. The point with coordinates $(0,\vec{0})$ is left invariant by the infinitesimal transformation generated by $M_{\mu\nu}$. The transformation generated by the boost M_{n0} followed by a one generated by the translations P_i allows to reach an arbitrary point with coordinates $(\tau,\vec{x})$. So we now have

$$\Phi(\tau,\vec{x}) = e^{-i\vec{x}\cdot\vec{P}} e^{i\tau M_{n0}} \Phi(0,\vec{0}) e^{-i\tau M_{n0}} e^{i\vec{x}\cdot\vec{P}}\,, \tag{B.4}$$

with

$$[\, M_{\mu\nu},\, \Phi(0,\vec{0})\,] = 0\,. \tag{B.5}$$

We have already solved this equation in eq.(8.17). A convenient basis of the UIR is formed by the eigenvectors $|\,\vec{p}\,\rangle$ of P_i. The action of the translations is diagonal on this basis. The explicit writing of eq.(B.4) amounts to calculate $\langle\,\vec{p}\,|\Psi_0\rangle$ and $e^{i\tau M_{n0}}|\,\vec{p}\,\rangle$. Since we have already the decomposition of $|\Psi_0\rangle$ on the position eigenstates $|\,\vec{\zeta}\,\rangle$ it is sufficient to calculate the connector $\langle\,\vec{\zeta}\,|\,\vec{p}\,\rangle$. Using the expression of the generators given in eq.(8.4), a direct calculation gives

$$\langle\,\vec{\zeta}\,|\,\vec{p}\,\rangle = 2^{-\frac{n}{2}}\pi^{-\frac{n-1}{2}}\,(1-\zeta_n)^{i\mu-\frac{n-1}{2}}\exp\left(i\frac{\vec{p}\cdot\vec{\zeta}'}{1-\zeta_n}\right). \tag{B.6}$$

In this equation, we have introduced a new notation. The vector $\vec{\zeta}$ lives on the $(n-1)$-dimensional sphere. It can thus be described by the following n coordinates $(\vec{\zeta}',\zeta_n)$ where $\vec{\zeta}'$ lives, like $\vec{p}$, in a $(n-1)$-dimensional Euclidean space.

The action of the boost $e^{i\tau M_{n0}}$ on $|\,\vec{p}\,\rangle$ can also be calculated using eq.(8.6, 8.7). The result is the simple expression:

$$e^{i\tau M_{n0}}|\,\vec{p}\,\rangle = e^{-(i\mu-\frac{n-1}{2})\tau}|\,e^{\tau}\vec{p}\,\rangle\,. \tag{B.7}$$

Finally we obtain the following expansion of the field operator:

$$\Phi(\tau,\vec{x}) = \int d^{n-1}\vec{p}\,a^{\dagger}(\vec{p})\,e^{i\vec{p}\cdot\vec{x}}e^{-(i\mu+\frac{n-1}{2})\tau}\langle\,e^{-\tau}\vec{p}|\Psi_0\rangle + \text{h.c.} \tag{B.8}$$

The expressions (8.17) and (B.6) for the components of $|\Psi_0\rangle$ and $|\,\vec{p}\,\rangle$ allow to calculate the amplitude $\langle\,\vec{p}|\Psi_0\rangle$ as

$$\begin{aligned}
\langle\,\vec{p}|\Psi_0\rangle &= \frac{1}{\sqrt{2}}\Gamma\left(i\mu-\frac{n-3}{2}\right)|\,\vec{p}\,|^{-i\mu} \\
&\quad\times\left[A\left\{\cos\left(\frac{n-3}{2}\pi\right)J_{-i\mu}(|\,\vec{p}\,|)+\sin\left(\frac{n-3}{2}\pi\right)Y_{-i\mu}(|\,\vec{p}\,|)\right\}+B\,J_{i\mu}(|\,\vec{p}\,|)\right] \\
&= \frac{1}{\sqrt{2}}\Gamma\left(i\mu-\frac{n-3}{2}\right)|\,\vec{p}\,|^{-i\mu}\sinh\left(\pi\mu+i\pi\frac{n-1}{2}\right)\left[\,C\,H^{(1)}_{i\mu}(|\,\vec{p}\,|)-D\,H^{(2)}_{i\mu}(|\,\vec{p}\,|)\,\right].
\end{aligned} \tag{B.9}$$

Notice that the $\tau\to-\infty$ limit is the same as the $|\,\vec{p}\,|\to\infty$ limit. The BD vacuum which is characterized by positive conformal frequencies in the high momentum limit here coincides with the *in* vacuum with positive conformal frequencies in the $\tau\to-\infty$ limit. The *out* vacuum is, as expected, the same one we got using the global sections. The various vacua can be straightforwardly obtained using the asymptotic behavior of the Bessel functions [39]:

$$
\begin{aligned}
&J_{i\mu}(z) \underset{z\to 0}{\approx} \frac{1}{\Gamma(i\mu+1)}\left(\frac{z}{2}\right)^{i\mu}, \\
&Y_{i\mu}(z) \underset{z\to 0}{\approx} -iH^{(1)}_{i\mu}(z) \underset{z\to 0}{\approx} iH^{(2)}_{i\mu}(z) \underset{z\to 0}{\approx} -\frac{\Gamma(i\mu)}{\pi}\left(\frac{z}{2}\right)^{-i\mu}, \\
&H^{(1)}_{i\mu}(z) \underset{z\to\infty}{\approx} \sqrt{\frac{2}{\pi z}}e^{\frac{\pi\mu}{2}}e^{i\left(z-\frac{\pi}{4}\right)}, \quad H^{(2)}_{i\mu}(z) \underset{z\to\infty}{\approx} \sqrt{\frac{2}{\pi z}}e^{-\frac{\pi\mu}{2}}e^{-i\left(z-\frac{\pi}{4}\right)}.
\end{aligned}
\tag{B.10}
$$

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